# Cross-sectional Dependence in Idiosyncratic Volatility* 

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#### Abstract

This paper introduces an econometric framework for analyzing cross-sectional dependence in the idiosyncratic volatilities of assets using high frequency data. We first consider the estimation of standard measures of dependence in the idiosyncratic volatilities such as covariances and correlations. Naive estimators of these measures are biased due to the use of the error-laden estimates of idiosyncratic volatilities. We provide bias-corrected estimators and the relevant asymptotic theory. Next, we introduce an idiosyncratic volatility factor model, in which we decompose the variation in idiosyncratic volatilities into two parts: the variation related to the systematic factors such as the market volatility, and the residual variation. Again, naive estimators of the decomposition are biased, and we provide bias-corrected estimators. We also provide the asymptotic theory that allows us to test whether the residual (non-systematic) components of the idiosyncratic volatilities exhibit cross-sectional dependence. We apply our methodology to the 30 Dow Jones Industrial Average components, and document strong cross-sectional dependence in their idiosyncratic volatilities. We consider two different sets of idiosyncratic volatility factors. We find that a single market volatility factor cannot fully account for the cross-sectional dependence in idiosyncratic volatilities, while this conclusion is reversed with additional industry volatility factors. For each model, we map out the network of dependencies in residual (non-systematic) idiosyncratic volatilities across all stocks.


Keywords: factor model, systematic risk, networks of risk, residual idiosyncratic volatility, high frequency data.

JEL Codes: C58, C22, C14, G11.

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## 1 Introduction

In a panel of assets, returns are generally cross-sectionally dependent. This dependence is usually modelled using the exposure of assets to some common return factors, such as the Fama-French factors. In this Return Factor Model (R-FM), the total volatility of an asset return can be decomposed into two parts: a component due to the exposure to the common return factors (the systematic volatility), and a residual component termed the Idiosyncratic Volatility (IdioVol). These two components of the volatility of returns are the most popular measures of the systematic risk and idiosyncratic risk of an asset.

Idiosyncratic Volatility is important in economics and finance for several reasons. For example, when arbitrageurs exploit the mispricing of an individual asset, they are exposed to the idiosyncratic risk of the asset and not the systematic risk (see, e.g., Campbell, Lettau, Malkiel, and Xu (2001)). ${ }^{1}$ Also, Idiosyncratic Volatility measures the exposure to the idiosyncratic risk in imperfectly diversified portfolios. The attention to IdioVols in empirical finance literature is exemplified by two IdioVol puzzles, started by Campbell, Lettau, Malkiel, and Xu (2001) and Ang, Hodrick, Xing, and Zhang (2006), each associated with its own follow-up literature. A recent observation is that the IdioVols seem to be strongly correlated in the cross-section of stocks, see, e.g., Connor, Korajczyk, and Linton (2006), Duarte, Kamara, Siegel, and Sun (2014), Herskovic, Kelly, Lustig, and Nieuwerburgh (2016), and Christoffersen, Fournier, and Jacobs (2018). Herskovic, Kelly, Lustig, and Nieuwerburgh (2016) argue this is due to a common IdioVol factor, which they relate to household risk. We note that the cross-sectional dependence in IdioVols is also important for option pricing, see Gourier (2016).

This paper provides an econometric framework for studying the cross-sectional dependence in the Idiosyncratic Volatilities using high frequency data. We show that naive estimators, such as covariances and correlations of estimated IdioVols used by several empirical studies, are substantially biased. The bias arises due to the use of error-laden estimates of IdioVols. We provide the bias-corrected estimators.

To study Idiosyncratic Volatilities, we introduce the Idiosyncratic Volatility Factor Model (IdioVol-FM). Just like a Return Factor Model, R-FM, such as the Fama-French model, decomposes returns into common and idiosyncratic returns, the IdioVol-FM decomposes the IdioVols into sys-

[^1]tematic and residual (non-systematic) components. The IdioVol factors may or may not be related to the return factors. The IdioVol factors can include the volatility of the return factors, or, more generally, (possibly non-linear) transformations of the spot covariance matrices of any observable variables, such as the average variance and average correlation factors of Chen and Petkova (2012). We propose bias-corrected estimators of the components of the IdioVol-FM model.

We provide the asymptotic theory for this model. For example, it allows us to test whether the residual (non-systematic) components of the IdioVols exhibit cross-sectional dependence. This allows us to identify the network of dependencies in the residual IdioVols across stocks.

Our bias-corrected estimators and inference results are an application of a new asymptotic theory that we develop for general estimators of quadratic covariation of vector-valued transformations of spot covariance matrices. This theoretical contribution is of its own interest. An example of alternative applications is the study of cross-sectional dependence of asset betas. Two features make the development of this asymptotic theory difficult. First, preliminary estimation of volatility results in first-order biases even in the special case of quadratic variation of the volatility one stock without any transformations, as in Vetter (2015). Second, we consider general nonlinear functionals in multivariate settings, which substantially complicates the analysis.

Throughout the paper, we use factors that are specified by the researcher. An example of our Return Factor Model is the so-called Fama-French factor model, which has three observable factors, or the CAPM, which has one observable factor (the market portfolio return). An example of our IdioVol factors is the market volatility, which can be estimated from the market index. Thus, our setup is different from settings such as PCA where factors are identified from the cross-section of the assets studied. The treatment of the latter case adds an additional layer of complexity to the model and is beyond the scope of the current paper.

We apply our methodology to high-frequency data on the 30 Dow Jones Industrial Average components. We study the IdioVols with respect to two models for asset returns: the CAPM and the three-factor Fama-French model. ${ }^{2}$ In both cases, the average pairwise correlation between the IdioVols is high (0.55). We verify that this dependence cannot be explained by the missing return factors. This confirms the recent findings of Herskovic, Kelly, Lustig, and Nieuwerburgh (2016) who use low frequency (daily and monthly) return data. We then consider the IdioVol-FM. We use two sets of IdioVol factors: the market volatility alone and the market volatility together with volatilities of nine industry ETFs. With the market volatility as the only IdioVol factor, the average

[^2]pairwise correlation between residual (non-systematic) IdioVols is substantially lower (0.25) than between the total IdioVols. We find that a single market volatility factor is not able to fully explain the cross-sectional dependence in IdioVols, while this conclusion is reversed for the richer IdioVolFM with industry volatility factors. For each model, we map out the network of dependencies in residual IdioVols across all stocks.

This paper analyzes cross-sectional dependence in Idiosyncratic Volatilities. This should not be confused with the analysis of cross-sectional dependence in total and idiosyncratic returns. A growing number of papers study the latter question using high frequency data. These date back to the analysis of realized covariances and their transformations, see, e.g., Barndorff-Nielsen and Shephard (2004) and Andersen, Bollerslev, Diebold, and Wu (2006). A continuous-time factor model for asset returns with observable return factors was first studied in Mykland and Zhang (2006). Various return factor models with observable factors have been studied by, among others, Bollerslev and Todorov (2010), Fan, Furger, and Xiu (2016), Li, Todorov, and Tauchen (2017a,b), and Aït-Sahalia, Kalnina, and Xiu (2020). Emerging literature also studies the cross-sectional dependence in returns using high-frequency data and latent return factors, see Aït-Sahalia and Xiu (2019, 2017) and Pelger (2019, 2020). Importantly, the models in the above papers are silent on the cross-sectional dependence structure in the IdioVols.

The Realized Beta GARCH model of Hansen, Lunde, and Voev (2014) imposes a structure on the cross-sectional dependence in IdioVols. This structure is tightly linked with the Return Factor Model parameters, whereas our stochastic volatility framework allows separate specification of the return factors and the IdioVol factors. ${ }^{3}$

Our inference theory is related to several results in the existing literature. First, as mentioned above, we generalize the result of Vetter (2015). Jacod and Rosenbaum (2013, 2015), Li, Todorov, and Tauchen (2016) and Li, Liu, and Xiu (2019) estimate integrated functionals of volatilities, which includes Idiosyncratic Volatilities. The latter problem is simpler than the problem of the current paper in the sense that $\sqrt{n}$-consistent estimation is possible, and the estimators are consistent without a bias correction (see Section 3.1 for details). In the literature on the estimation of the leverage effect, preliminary estimation of volatility also creates a bias, which also needs to be corrected to achieve consistency, see Aït-Sahalia, Fan, and Li (2013), Aït-Sahalia, Fan, Laeven, Wang, and Yang (2017), Kalnina and Xiu (2017) and Wang and Mykland (2014). The biases due

[^3]to preliminary estimation of volatility can be made theoretically negligible when an additional, long-span, asymptotic approximation is used. This requires the assumption that the frequency of observations is high enough compared to the time span, see, e.g., Corradi and Distaso (2006), Bandi and Renò (2012), Li and Patton (2018), and Kanaya and Kristensen (2016).

In the empirical section, we define a network of dependencies using (functions of) quadratic covariations of IdioVols. This approach can be compared with the network connectedness measures of Diebold and Yilmaz (2014). The latter measures are based on forecast error variance decompositions from vector autoregressions. They capture co-movements in forecast errors. In contrast, we assume a general semimartingale setting, and our framework captures realized co-movements in Idiosyncratic Volatilities, while accounting for the measurement errors in these volatilities.

The remainder of the paper is organized as follows. Section 2 introduces the model and the quantities of interest. Section 3 describes the identification and estimation. Section 4 presents the asymptotic properties of our estimators. Section 5 uses high-frequency stock return data to study the cross-sectional dependence in IdioVols using our framework. Section 6 contains Monte Carlo simulations. The Appendix contains all proofs and additional figures.

## 2 Model and Quantities of Interest

We first describe a general Factor Model for the Returns (R-FM), which allows us to define the Idiosyncratic Volatility. We then introduce the Idiosyncratic Volatility Factor Model (IdioVol-FM). In this framework, we proceed to define the cross-sectional measures of dependence between the total IdioVols, as well as the residual IdioVols, which take into account the dependence induced by the IdioVol factors.

Suppose we have $(\log )$ prices on $d_{S}$ assets such as stocks, $S_{t}=\left(S_{1, t}, \ldots, S_{d_{S}, t}\right)^{\top}$, and on $d_{F}$ observable factors, $F_{t}=\left(F_{1, t}, \ldots, F_{d_{F}, t}\right)^{\top}$. We stack them into the $d$-dimensional process $Y_{t}=\left(S_{1, t}, \ldots, S_{d_{S}, t}, F_{1, t}, \ldots, F_{d_{F}, t}\right)^{\top}$ where $d=d_{S}+d_{F}$. The observable factors $F_{1}, \ldots F_{d_{F}}$ are used in the R-FM model below. We assume that all observable variables jointly follow an Itô semimartingale, i.e., $Y_{t}$ follows

$$
\begin{equation*}
Y_{t}=Y_{0}+\int_{0}^{t} b_{s} d s+\int_{0}^{t} \sigma_{s} d W_{s}+J_{t} \tag{1}
\end{equation*}
$$

where $W$ is a $d^{W}$-dimensional Brownian motion $\left(d^{W} \geq d\right), C_{t}=\sigma_{t} \sigma_{t}^{\top}$ is the spot covariance process,
and $J_{t}$ denotes a finite variation jump process. The spot covariance matrix process $C_{t}$ of $Y_{t}$ is a continuous Itô semimartingale, ${ }^{4}$

$$
\begin{equation*}
C_{t}=C_{0}+\int_{0}^{t} \widetilde{b}_{s} d s+\int_{0}^{t} \widetilde{\sigma}_{s} d W_{s} \tag{2}
\end{equation*}
$$

We refer to the $\left(C_{t}\right)_{a, b}$ element of the matrix $C_{t}$ as $C_{a b, t}$. For convenience, we also use the alternative notation $C_{U V, t}$ to refer to the spot covariance between two elements $U$ and $V$ of $Y$, and $C_{U, t}$ to refer to $C_{U U, t}$.

We assume a standard continuous-time factor model for the asset returns.
Definition (Factor Model for Returns, R-FM). For all $0 \leq t \leq T$ and $j=1, \ldots, d_{S},{ }^{5}$

$$
\begin{align*}
d S_{j, t} & =\beta_{j, t}^{\top} d F_{t}^{c}+\tilde{\beta}_{j, t}^{\top} d F_{t}^{d}+d Z_{j, t} \quad \text { with }  \tag{3}\\
{\left[Z_{j}, F\right]_{t} } & =0 .
\end{align*}
$$

In the above, $d Z_{j, t}$ is the idiosyncratic return of stock $j$. The superscripts $c$ and $d$ indicate the continuous and jump part of the processes, so that $\beta_{j, t}$ and $\tilde{\beta}_{j, t}$ are the continuous and jump factor loadings. For example, the $k$-th component of $\beta_{j, t}$ corresponds to the time-varying loading of the continuous part of the return on stock $j$ to the continuous part of the return on the $k$-th factor. We set $\beta_{t}=\left(\beta_{1, t}, \ldots, \beta_{d_{S}, t}\right)^{\top}$ and $Z_{t}=\left(Z_{1, t}, \ldots, Z_{d_{S}, t}\right)^{\top}$.

We do not need the return factors $F_{t}$ to be the same across assets to identify the model, but without loss of generality, we keep this structure as it is standard in empirical finance. These return factors are assumed to be observable, which is also standard. For example, in the empirical application, we use two sets of return factors: the market portfolio and the three Fama-French factors, which are constructed in Aït-Sahalia, Kalnina, and Xiu (2020).

A continuous-time factor model for returns with observable factors was originally studied in Mykland and Zhang (2006) in the case of one factor and in the absence of jumps. A burgeoning literature uses related models to study the cross-sectional dependence of total and/or idiosyncratic

[^4]returns. However, this literature does not consider the cross-sectional dependence in the IdioVols.
We define the idiosyncratic Volatility (IdioVol) to be the spot volatility of $Z_{j, t}$ and denote it by $C_{Z j, t}$. Notice that R-FM in (3) implies that the factor loadings $\beta_{t}$ as well as the IdioVols are functions of the total spot covariance matrix $C_{t}$. In particular, the vector of factor loadings satisfies
\[

$$
\begin{equation*}
\beta_{j t}=\left(C_{F, t}\right)^{-1} C_{F S j, t}, \tag{4}
\end{equation*}
$$

\]

for $j=1, \ldots, d_{S}$, where $C_{F, t}$ denotes the spot covariance matrix of the factors $F$, which is the lower $d_{F} \times d_{F}$ sub-matrix of $C_{t}$; and $C_{F S j, t}$ denotes the covariance of the factors and the $j^{t h}$ stock, which is a vector consisting of the last $d_{F}$ elements of the $j^{t h}$ column of $C_{t}$. The IdioVol of stock $j$ is then also a function of the total spot covariance matrix $C_{t}$,

$$
\begin{equation*}
\underbrace{C_{Z j, t}}_{\text {IdioVol of stock } \mathrm{j}}=\underbrace{C_{Y j, t}}_{\text {total volatility of stock } \mathrm{j}}-\left(C_{F S j, t}\right)^{\top}\left(C_{F, t}\right)^{-1} C_{F S j, t} . \tag{5}
\end{equation*}
$$

By the Itô lemma, (4) and (5) imply that factor loadings and IdioVols are also Itô semimartingales with characteristics that are functions of $C_{t}$.

We now introduce the Idiosyncratic Volatility Factor model (IdioVol-FM). In IdioVol-FM, the cross-sectional dependence in the IdioVol shocks can be potentially explained by certain IdioVol factors. We assume the IdioVol factors are given smooth functions of the matrix $C_{t}$. In the empirical application, we use the market volatility as the IdioVol factor, which has been used in Herskovic, Kelly, Lustig, and Nieuwerburgh (2016) and Gourier (2016); we discuss other possibilities below.

Definition (Idiosyncratic Volatility Factor Model, IdioVol-FM). For all $0 \leq t \leq T$ and $j=1, \ldots, d_{S}$, the Idiosyncratic Volatility $C_{Z j}$ follows,

$$
\begin{align*}
d C_{Z j, t} & =\gamma_{Z j}^{\top} d \Pi_{t}+d C_{Z j, t}^{r e s i d} \text { with }  \tag{6}\\
{\left[C_{Z j}^{r e s i d}, \Pi\right]_{t} } & =0,
\end{align*}
$$

where $\Pi_{t}=\left(\Pi_{1 t}, \ldots, \Pi_{d_{\Pi} t}\right)$ is a $\mathbb{R}^{d_{\Pi}}$-valued vector of IdioVol factors, which satisfy

$$
\begin{equation*}
\Pi_{k t}=\Pi_{k}\left(C_{t}\right) \tag{7}
\end{equation*}
$$

with the function $\Pi_{k}(\cdot)$ being three times continuously differentiable for $k=1, \ldots, d_{\Pi}$.
We call the residual term $C_{Z j, t}^{r e s i d}$ the residual IdioVol of asset $j$. Our assumptions imply that
the components of the IdioVol-FM, $C_{Z j, t}, \Pi_{t}$ and $C_{Z j, t}^{r e s i d}$, are continuous Itô semimartingales. We remark that both the dependent variable and the regressors in our IdioVol-FM are not directly observable and have to be estimated, and our asymptotic theory takes that into account. As will see in Section 3, this preliminary estimation implies that the naive estimators of all the dependence measures defined below are biased. One of the contributions of this paper is to quantify this bias and provide the bias-corrected estimators for all the quantities of interest.

The class of IdioVol factors permitted by our theory is rather wide as it includes general nonlinear transforms of the spot covariance matrix process $C_{t}$. For example, IdioVol factors can be linear combinations of the total volatilities of stocks, see, e.g., the average variance factor of Chen and Petkova (2012). Other examples of IdioVol factors are linear combinations of the IdioVols, such as the equally-weighted average of the IdioVols, which Herskovic, Kelly, Lustig, and Nieuwerburgh (2016) denote by the "CIV". The IdioVol factors can also be the volatilities of any other observable processes.

Having specified our econometric framework, we now provide the definitions of some natural measures of dependence for (residual) IdioVols. Their estimation is discussed in Section 3.

Before considering the effect of IdioVol factors by using the IdioVol-FM decomposition, one may be interested in quantifying the dependence between the IdioVols of two stocks $j$ and $s$. A natural measure of dependence is the quadratic-covariation based correlation between the two IdioVol processes over a given time period $[0, T]$,

$$
\begin{equation*}
\operatorname{Corr}\left(C_{Z j}, C_{Z s}\right)=\frac{\left[C_{Z j}, C_{Z s}\right]_{T}}{\sqrt{\left[C_{Z j}, C_{Z j}\right]_{T}} \sqrt{\left[C_{Z s}, C_{Z s}\right]_{T}}} . \tag{8}
\end{equation*}
$$

Alternatively, one may consider the quadratic covariation $\left[C_{Z j}, C_{Z s}\right]_{T}$ without any normalization. In Section 4.4, we use the estimator of the latter quantity to test for the presence of cross-sectional dependence in IdioVols.

To measure the residual cross-sectional dependence between the IdioVols of two stocks after accounting for the effect of the IdioVol factors, we use again the quadratic-covariation based correlation,

$$
\begin{equation*}
\operatorname{Corr}\left(C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right)=\frac{\left[C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T}}{\sqrt{\left[C_{Z j}^{\text {resid }}, C_{Z j}^{\text {resid }}\right]_{T}} \sqrt{\left[C_{Z s}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T}}} \tag{9}
\end{equation*}
$$

In Section 4.4, we use the quadratic covariation between the two residual IdioVol processes
$\left[C_{Z j}^{r e s i d}, C_{Z s}^{r e s i d}\right]_{T}$ without normalization for testing purposes.
We want to capture how well the IdioVol factors explain the time variation of IdioVols of the $j^{\text {th }}$ asset. For this purpose, we use the quadratic-covariation based analog of the coefficient of determination. For $j=1, \ldots, d_{S}$,

$$
\begin{equation*}
R_{Z j}^{2, \text { IdioVol-FM }}=\frac{\gamma_{Z j}^{\top}[\Pi, \Pi]_{T} \gamma_{Z j}}{\left[C_{Z j}, C_{Z j}\right]_{T}} \tag{10}
\end{equation*}
$$

It is interesting to compare the correlation measure between IdioVols in equation (8) with the correlation between the residual parts of IdioVols in (9). We consider their difference,

$$
\begin{equation*}
\operatorname{Corr}\left(C_{Z j}, C_{Z s}\right)-\operatorname{Corr}\left(C_{Z j}^{r e s i d}, C_{Z s}^{r e s i d}\right) \tag{11}
\end{equation*}
$$

to see how much of the dependence between IdioVols can be attributed to the IdioVol factors. In practice, if we compare assets that are known to have positive covolatilities (typically, stocks have that property), another useful measure of the common part in the overall covariation between IdioVols is the following quantity,

$$
\begin{equation*}
Q_{Z j, Z s}^{\text {IdioVol-FM }}=\frac{\gamma_{Z j}^{\top}[\Pi, \Pi]_{T} \gamma_{Z s}}{\left[C_{Z j}, C_{Z s}\right]_{T}} \tag{12}
\end{equation*}
$$

This measure is bounded by 1 if the covariations between residual IdioVols are nonnegative and smaller than the covariations between IdioVols, which is what we find for every pair in our empirical application with high-frequency observations on stock returns.

We remark that our framework can be compared with the following null hypothesis studied in Li, Todorov, and Tauchen (2016), $H_{0}: C_{Z j, t}=a_{Z j}+\gamma_{Z j}^{\top} \Pi_{t}, 0 \leq t \leq T$. This $H_{0}$ implies that the IdioVol is a deterministic function of the factors, which does not allow for an error term. In particular, this null hypothesis implies $R_{Z j}^{2, \text { IdioVol-FM }}=1$.

## 3 Estimation

As we show below, the quantities of interest in Section 2 can be expressed in terms of quadratic covariation between two functions of the spot covariance matrix $C_{t}$,

$$
\begin{equation*}
[H(C), G(C)]_{T} \tag{13}
\end{equation*}
$$

Section 3.1 proposes estimators of this general functional, and Section 3.2 explains how to use these formulas to obtain estimators of the quantities of interest in Section 2.

### 3.1 Estimation of a General Functional

This section proposes estimators of the quadratic covariation between two functions of the spot covariance matrix $[H(C), G(C)]_{T}$, where $H$ and $G$ are given real-valued smooth functions. Recall that $C_{t}$ is the spot covariance matrix of the observable variables, see equations (1)-(2).

Suppose we have discrete observations on $Y_{t}$ over an interval $[0, T]$. Denote by $\Delta_{n}$ the distance between observations. It is well known that we can estimate the spot covariance matrix $C_{t}$ at time $(i-1) \Delta_{n}$ with a local truncated realized volatility estimator,

$$
\begin{equation*}
\widehat{C}_{i \Delta_{n}}=\frac{1}{k_{n} \Delta_{n}} \sum_{m=0}^{k_{n}-1}\left(\Delta_{i+m}^{n} Y\right)\left(\Delta_{i+m}^{n} Y\right)^{\top} 1_{\left\{\left\|\Delta_{i+m}^{n} Y\right\| \leq \chi \Delta_{n}^{\varpi}\right\}}, \tag{14}
\end{equation*}
$$

where $\Delta_{i}^{n} Y=Y_{i \Delta_{n}}-Y_{(i-1) \Delta_{n}}$ and where $k_{n}$ is the number of observations in a local window. ${ }^{6}$ We refer to the $\left(\widehat{C}_{i \Delta_{n}}\right)_{a, b}$ element of the matrix $\widehat{C}_{i \Delta_{n}}$ as $\widehat{C}_{a b, i \Delta_{n}}$.

If $C_{i \Delta_{n}}$ was observed, $[H(C), G(C)]_{T}$ could be estimated by the realized covariance between $G\left(C_{i \Delta_{n}}\right)$ and $H\left(C_{i \Delta_{n}}\right)$, which is the sample analog of the definition of $[H(C), G(C)]_{T}$. However, we do not observe $C_{i \Delta_{n}}$. If we replace it with $\widehat{C}_{i \Delta_{n}}$ in (14), we obtain the plug-in estimator

$$
\begin{equation*}
\left[H(\widehat{C), G}(C)]_{T}^{\text {Naive }}=\frac{1}{k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\left(H\left(\widehat{C}_{\left(i+k_{n}\right) \Delta_{n}}\right)-H\left(\widehat{C}_{i \Delta_{n}}\right)\right)\left(G\left(\widehat{C}_{\left(i+k_{n}\right) \Delta_{n}}\right)-G\left(\widehat{C}_{i \Delta_{n}}\right)\right)\right) .\right. \tag{15}
\end{equation*}
$$

However, it turns out that due to the measurement errors in $\widehat{C}_{i \Delta_{n}}$, this estimator is inconsistent.
We propose two estimators for the general quantity $[H(C), G(C)]_{T}$. The first is a bias-corrected sample analog of the definition of quadratic covariation between two Itô processes,

$$
\begin{align*}
{\left[H(\widehat{C), G}(C)]_{T}^{A N}\right.} & =\frac{3}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\left(H\left(\widehat{C}_{\left(i+k_{n}\right) \Delta_{n}}\right)-H\left(\widehat{C}_{i \Delta_{n}}\right)\right)\left(G\left(\widehat{C}_{\left(i+k_{n}\right) \Delta_{n}}\right)-G\left(\widehat{C}_{i \Delta_{n}}\right)\right)\right. \\
& \left.-\frac{2}{k_{n}} \sum_{g, h, a, b=1}^{d}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i \Delta_{n}}\right)\left(\widehat{C}_{g a, i \Delta_{n}} \widehat{C}_{g b, i \Delta_{n}}+\widehat{C}_{g b, i \Delta_{n}} \widehat{C}_{h a, i \Delta_{n}}\right)\right) \tag{16}
\end{align*}
$$

[^5]Our second estimator is based on the following equality, which follows by the Itô lemma,

$$
\begin{equation*}
[H(C), G(C)]_{T}=\sum_{g, h, a, b=1}^{d} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right) \bar{C}_{t}^{g h, a b} d t, \tag{17}
\end{equation*}
$$

where $\bar{C}_{t}^{g h, a b}$ denotes the covariation between the volatility processes $C_{g h, t}$ and $C_{a b, t}$. The quantity is thus a non-linear functional of the spot covariance and spot volatility of volatility matrices. Our second estimator is a bias-corrected version of the sample counterpart of the "linearized" expression in (17),

$$
\begin{align*}
& {\left[H(\widehat{C), G}(C)]_{T}^{L I N}=\frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i \Delta_{n}}\right) \times\right.}  \tag{18}\\
& \left(\left(\widehat{C}_{g h,\left(i+k_{n}\right) \Delta_{n}}-\widehat{C}_{g h, i \Delta_{n}}\right)\left(\widehat{C}_{a b,\left(i+k_{n}\right) \Delta_{n}}-\widehat{C}_{a b, i \Delta_{n}}\right)-\frac{2}{k_{n}}\left(\widehat{C}_{g a, i \Delta_{n}} \widehat{C}_{g b, i \Delta_{n}}+\widehat{C}_{g b, i \Delta_{n}} \widehat{C}_{h a, i \Delta_{n}}\right)\right) .
\end{align*}
$$

We now provide the intuition for the bias terms. If we had observations on $C_{i \Delta_{n}}$, the estimators of $[H(C), G(C)]_{T}$ would not need any bias-correction terms. It is useful to think of $\widehat{C}_{i \Delta_{n}}$ as an estimator of integrated volatility matrix, $\widehat{C}_{i \Delta_{n}}=\frac{1}{k_{n} \Delta_{n}} \int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}} C_{s} d s+U_{i \Delta_{n}}$, where $U_{i \Delta_{n}}$ is the estimation error. The first part of the bias-correction in (16) and (18) is an additive term

$$
\begin{equation*}
-\frac{3}{k_{n}^{2}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\sum_{g, h, a, b=1}^{d}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i \Delta_{n}}\right)\left(\widehat{C}_{g a, i \Delta_{n}} \widehat{C}_{g b, i \Delta_{n}}+\widehat{C}_{g b, i \Delta_{n}} \widehat{C}_{h a, i \Delta_{n}}\right)\right) . \tag{19}
\end{equation*}
$$

This term arises because of the estimation error $U_{i \Delta_{n}}$. Intuitively, estimation of, e.g., variance of functionals of $C_{i \Delta_{n}}$ by variance of functionals of $\widehat{C}_{i \Delta_{n}}$ overestimates it due to the additional variability of $U_{i \Delta_{n}}$. In particular, one can show that the additive bias-correction term in (19) is, up to a scale factor, an estimator of the asymptotic covariance between the estimators of $\int_{0}^{T} H\left(C_{t}\right) d t$ and $\int_{0}^{T} G\left(C_{t}\right) d t$.

The second part of the bias-correction in (16) and (18) is the multiplicative correction factor $3 / 2$. This correction factor is needed because of a smoothing bias that arises due to the replacement of $C_{i \Delta_{n}}$ by $\frac{1}{\Delta_{n}} \int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}} C_{s} d s$. To gain some intuition, consider the special case of $d=1$ and $H(\cdot)=G(\cdot)=\cdot$. Suppose we had observations on $\frac{1}{\Delta_{n}} \int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}} C_{s} d s$. The $i^{t h}$ summand in the naive estimator of $[C, C]_{T}$ would be

$$
\begin{equation*}
\left(\int_{\left(i+k_{n}\right) \Delta_{n}}^{\left(i+2 k_{n}\right) \Delta_{n}} C_{s} d s-\int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}} C_{s} d s\right)^{2}=\left(\int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}}\left(C_{s+\Delta_{n} k_{n}}-C_{s}\right) d s\right)^{2} \tag{20}
\end{equation*}
$$

divided by $\Delta_{n}^{2} k_{n}^{3}$. Consider the weights that the integral $\int_{i \Delta_{n}}^{\left(i+k_{n}\right) \Delta_{n}}\left(C_{s+\Delta_{n} k_{n}}-C_{s}\right) d s$ puts on $\Delta_{n}$-increments of the volatility $C_{t}$ : these weights are triangular, i.e., $\left(\Delta_{n} k_{n}-\left|\Delta_{n} k_{n}+i \Delta_{n}-s\right|\right) I\left\{s \in\left[i \Delta_{n},\left(i+2 k_{n}\right) \Delta_{n}\right]\right\}$. One can show that the squared integral in (20) is proportional to the integral of the squared triangular weights, $\frac{1}{\left(\Delta_{n} k_{n}\right)^{3}} \int_{i \Delta_{n}}^{\left(2 k_{n}+i\right) \Delta_{n}}\left(\Delta_{n} k_{n}-\left|\Delta_{n} k_{n}+i \Delta_{n}-s\right|\right)^{2} d s$. The latter integral equals $\frac{2}{3}$, hence the estimator needs a multiplicative correction factor $\frac{3}{2}$.

It is useful to describe how our asymptotic theory is related to the earlier work of Jacod and Rosenbaum (2013) and Jacod and Rosenbaum (2015), JR13 and JR15 henceforth. The parameter of interest in those two papers is the integrated functional of volatility $\int_{0}^{T} H\left(C_{t}\right) d t$, which is different from the quadratic covariation $[H(C), G(C)]_{T}$. The estimation of integrated functionals of volatility is simpler in a number of ways. First, the naive plug-in estimators of $\int_{0}^{T} H\left(C_{t}\right) d t$ are consistent, and a bias-correction is only needed to derive the asymptotic distribution. In contrast, the naive plugin estimators of $[H(C), G(C)]_{T}$ are inconsistent without a bias-correction. Second, the estimators of $\int_{0}^{T} H\left(C_{t}\right) d t$ have a rate of convergence $n^{1 / 2}$, while our estimators have a rate of convergence $n^{1 / 4}$. Third, our analysis requires a proof strategy that is different from JR13\&15. To obtain the asymptotic distribution of the estimators of $\int_{0}^{T} H\left(C_{t}\right) d t$, JR13\&15 approximate volatility to be piecewise constant. This approximation does not work in our case, which substantially complicates the proof. We also remark on another connection with JR15. One of the higher-order bias terms in JR15 is of the form $[H(C), H(C)]_{T}$. In the special case $H(\cdot)=G(\cdot)$, aside from a scale factor and the end-effects, our LIN estimator in equation (18) coincides with their estimator. Note that JR15 only establish consistency of their estimator, which is all they need to implement their bias correction. In contrast, our analysis considers estimators of a general quadratic covariation, derives their asymptotic distributions, and proposes consistent estimators of the asymptotic variances.

Our two estimators, AN in equation (16) and LIN in (18), are identical when $H$ and $G$ are linear, for example, when estimating the covariation between two volatility processes. In the univariate case $d=1$, when $H(\cdot)=G(\cdot)=\cdot$, our estimator coincides with the volatility of volatility estimator of Vetter (2015). Our contribution is the development of the asymptotic theory for general nonlinear functionals and allowing $d>1$.

How do the two estimators, AN and LIN, compare (when they are not identical)? In the next section, we show that they have the same asymptotic distribution. In our Monte Carlo experiments in Section 6, LIN estimator somewhat outperforms AN estimator. We leave the theoretical analysis
of asymptotic higher-order properties of these estimators to future research.

### 3.2 Estimation in R-FM and IdioVol-FM models

In this section, we explain how to use formulas in equations (16) and (18) to obtain estimators for the objects of interest in Section 2, see equations (6)-(12). In particular, each of these objects of interest,

$$
\begin{align*}
& {\left[C_{Z j}, C_{Z s}\right]_{T}, \operatorname{Corr}\left(C_{Z j}, C_{Z s}\right), \gamma_{Z_{j}},\left[C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T},} \\
& \operatorname{Corr}\left(C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right), Q_{Z j, Z s}^{\text {IdioVol-FM }}, \text { and } R_{Z j}^{2, \text { Idio Vol-FM }}, \tag{21}
\end{align*}
$$

for $j, s=1, \ldots, d_{S}$, can be written as

$$
\begin{equation*}
\varphi\left(\left[H_{1}(C), G_{1}(C)\right]_{T}, \ldots,\left[H_{\kappa}(C), G_{\kappa}(C)\right]_{T}\right) \tag{22}
\end{equation*}
$$

for some smooth, real-valued functions $\varphi, H_{r}, G_{r}, r=1, \ldots, \kappa$. Each element in (22) is of the form $\left[H_{r}(C), G_{r}(C)\right]_{T}$, i.e., it is a quadratic covariation between functions of $C_{t}$, and hence can be estimated using the estimators proposed in Section 3.1.

We start by discussing the first quantity in (21), which is the quadratic covariation between $j^{\text {th }}$ and $s^{t h}$ IdioVol, $\left[C_{Z j}, C_{Z s}\right]_{T}$. It can be written as $[H(C), G(C)]_{T}$ if we choose $H\left(C_{t}\right)=C_{Z j, t}$ and $G\left(C_{t}\right)=C_{Z s, t}$. By equation (5), both $C_{Z j, t}$ and $C_{Z s, t}$ are functions of $C_{t}$.

Next, consider $\operatorname{Corr}\left(C_{Z j}, C_{Z s}\right)$ defined in equation (8). Correlation is a function of three quadratic covariations, each of which can be represented in the form $\left[H_{r}(C), G_{r}(C)\right]_{T}$. Therefore, $\operatorname{Corr}\left(C_{Z j}, C_{Z s}\right)$ is of the form of equation (22).

Note that IdioVol-FM implies

$$
\begin{align*}
\gamma_{Z j} & =\left([\Pi, \Pi]_{T}\right)^{-1}\left[\Pi, C_{Z j}\right]_{T}, \quad \text { and }  \tag{23}\\
{\left[C_{Z j}^{r e s i d}, C_{Z s}^{r e s i d}\right]_{T} } & =\left[C_{Z j}, C_{Z s}\right]_{T}-\gamma_{Z j}^{\top}[\Pi, \Pi]_{T} \gamma_{Z s} \tag{24}
\end{align*}
$$

for $j, s=1, \ldots, d_{S}$. Recall that $C_{Z j, t}, C_{Z s, t}$ and every element of $\Pi_{t}$ are given real-valued functions of $C_{t}$. Thus, the right-hand-sides of (23) and (24) have the form of equation (22), for a finite number of quantities of the form $\left[H_{r}(C), G_{r}(C)\right]_{T}$.

Finally, $\operatorname{Corr}\left(C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right), Q_{Z j, Z s}^{\text {IdioVol-FM }}$ and $R_{Z j}^{2, \text { IdioVol-FM }}$ are smooth functions of $\left[C_{Z j}^{\text {resid }}, C_{Z j}^{\text {resid }}\right]_{T},\left[C_{Z j}, C_{Z j}\right]_{T}, \gamma_{Z j}$, and $[\Pi, \Pi]_{T}$, each of which is of the form of equation (22),
and hence are themselves of the form of equation (22).

## 4 Asymptotic Properties

In this section, we first present the full list of assumptions for our asymptotic results. We then obtain the joint asymptotic distribution between the general functionals $\left[H_{r}(C), G_{r}(C)\right]_{T}$ for $r=1, \ldots, \kappa$ introduced in Section 3.1. We also develop estimators for the asymptotic variance-covariance matrix. The asymptotic distributions of the estimators of $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$ and other quantities of interest in Section 2 follow by the Delta method (see Section 3.2 for details). Finally, to illustrate the application of the general theory, we describe three statistical tests about the IdioVols, which we later implement in the empirical and Monte Carlo analysis.

### 4.1 Assumptions

Recall that the $d$-dimensional process $Y_{t}$ represents the $(\log )$ prices of stocks, $S_{t}$, and factors $F_{t}$.
Assumption 1. Suppose $Y$ is an Itô semimartingale on a filtered space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geq 0}, \mathbb{P}\right)$,

$$
\begin{equation*}
Y_{t}=Y_{0}+\int_{0}^{t} b_{s} d s+\int_{0}^{t} \sigma_{s} d W_{s}+\int_{0}^{t} \int_{E} \delta(s, z) \mu(d s, d z), \tag{25}
\end{equation*}
$$

where $W$ is a $d^{W}$-dimensional Brownian motion $\left(d^{W} \geq d\right)$ and $\mu$ is a Poisson random measure on $\mathbb{R}_{+} \times E$, with $E$ an auxiliary Polish space with intensity measure $\nu(d t, d z)=d t \otimes \lambda(d z)$ for some $\sigma$ finite measure $\lambda$ on $E$. The process $b_{t}$ is $\mathbb{R}^{d}$-valued optional, $\sigma_{t}$ is $\mathbb{R}^{d} \times \mathbb{R}^{d^{W}}$-valued, and $\delta=\delta(w, t, z)$ is a predictable $\mathbb{R}^{d}$-valued function on $\Omega \times \mathbb{R}_{+} \times E$. Moreover, $\left\|\delta\left(w, t \wedge \tau_{m}(w), z\right)\right\| \wedge 1 \leq \Gamma_{m}(z)$, for all ( $w, t, z$ ), where $\left(\tau_{m}\right)$ is a localizing sequence of stopping times and, for some $r \in[0,1 / 2)$, the function $\Gamma_{m}$ on $E$ satisfies $\int_{E} \Gamma_{m}(z)^{r} \lambda(d z)<\infty$. The spot volatility matrix of $Y$ is then defined as $C_{t}=\sigma_{t} \sigma_{t}^{\top}$. We assume that $C_{t}$ is a continuous Itô semimartingale, ${ }^{7}$

$$
\begin{equation*}
C_{t}=C_{0}+\int_{0}^{t} \widetilde{b}_{s} d s+\int_{0}^{t} \widetilde{\sigma}_{s} d W_{s} \tag{26}
\end{equation*}
$$

where $\widetilde{b}$ is $\mathbb{R}^{d} \times \mathbb{R}^{d}$-valued optional. $C_{t}$ takes values in the space $\mathcal{M}_{d}$ consisting of $d \times d$ positive definite matrices. For a sequence of convex compact subsets $\left(\mathcal{K}_{m}\right)_{m \geq 1}$ of $\mathcal{M}_{d}, C_{t} \in \mathcal{K}_{m}$ for all $t \leq \tau_{m}$.

[^6]With the above notation, the elements of the spot volatility of volatility matrix and spot covariation of the continuous martingale parts of $X$ and $c$ are defined as follows,

$$
\begin{equation*}
\bar{C}_{t}^{g h, a b}=\sum_{m=1}^{d^{W}} \widetilde{\sigma}_{t}^{g h, m} \widetilde{\sigma}_{t}^{a b, m}, \bar{C}_{t}^{\prime g, a b}=\sum_{m=1}^{d^{W}} \sigma_{t}^{g m} \widetilde{\sigma}_{t}^{a b, m} . \tag{27}
\end{equation*}
$$

We assume the following for the process $\widetilde{\sigma}_{t}$ :
Assumption 2. $\widetilde{\sigma}_{t}$ is a continuous Itô semimartingale with its characteristics satisfying the same requirements as that of $C_{t}$.

Assumption 1 is very general and nests most of the multivariate continuous-time models used in economics and finance. It allows for potential stochastic volatility and jumps in returns. Assumption 2 is required to obtain the asymptotic distribution of estimators of the quadratic covariation between functionals of the spot covariance matrix $C_{t}$. It is not needed to prove consistency. This assumption also appears in Wang and Mykland (2014), Vetter (2015), and Kalnina and Xiu (2017).

### 4.2 Asymptotic Distribution

We have seen in Section 3 that all quantities of interest in (21) are functions of multiple objects of the form $[H(C), G(C)]_{T}$. Therefore, if we can obtain a multivariate asymptotic distribution for a vector with elements of the form $[H(C), G(C)]_{T}$, the asymptotic distributions for all our estimators follow by the Delta method. The current section presents this asymptotic distribution.

Let $H_{1}, G_{1}, \ldots, H_{\kappa}, G_{\kappa}$ be given smooth real-valued functions. We are interested in the asymptotic behavior of vectors

$$
\begin{align*}
& \left(\left[H_{1}\left(\widehat{C), G_{1}}(C)\right]_{T}^{A N}, \ldots,\left[H_{\kappa}\left(\widehat{C), G_{\kappa}}(C)\right]_{T}^{A N}\right)^{\top}\right.\right. \text { and } \\
& \left(\left[H_{1}\left(\widehat{C), G_{1}}(C)\right]_{T}^{L I N}, \ldots,\left[H_{\kappa}\left(\widehat{C), G_{\kappa}}(C)\right]_{T}^{L I N}\right)^{\top}\right.\right. \tag{28}
\end{align*}
$$

The following theorem summarizes the joint asymptotic behavior of the estimators.
Theorem 1. Let $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}\right.$ denote either $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}^{A N}\right.$ or $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}^{L I N}\right.$ defined in equations (16) and (18), where $H_{r}$ and $G_{r}$ are three times differentiable real-valued functions, for $r=1, \ldots, \kappa$. Suppose Assumptions 1 and 2 hold. Fix $k_{n}=\theta \Delta_{n}^{-1 / 2}$ for some $\theta \in(0, \infty)$ and set
$3 / 4(2-r) \leq \varpi<\frac{1}{2}$. Then, as $\Delta_{n} \rightarrow 0$,

$$
\Delta_{n}^{-1 / 4}\left(\begin{array}{c}
{\left[H_{1}\left(\widehat{C), G_{1}}(C)\right]_{T}-\left[H_{1}(C), G_{1}(C)\right]_{T}\right.}  \tag{29}\\
\cdots \\
{\left[H_{\kappa}\left(\widehat{C), G_{\kappa}}(C)\right]_{T}-\left[H_{\kappa}(C), G_{\kappa}(C)\right]_{T}\right.}
\end{array}\right) \xrightarrow{L-s} M N\left(0, \Sigma_{T}\right)
$$

Let $\Sigma_{T}^{r, s}$ be the $\left(\Sigma_{T}\right)_{r, s}$ element of the $\kappa \times \kappa$ matrix $\Sigma_{T}$. We have

$$
\begin{aligned}
& \Sigma_{T}^{r, s}=\Sigma_{T}^{r, s,(1)}+\Sigma_{T}^{r, s,(2)}+\Sigma_{T}^{r, s,(3)}, \\
& \Sigma_{T}^{r, s,(1)}=\frac{6}{\theta^{3}} \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d} \int_{0}^{T}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(C_{s}\right)\right)\left[C_{t}(g h, j k) C_{t}(a b, l m)\right. \\
& \left.\quad+C_{t}(a b, j k) C_{t}(g h, l m)\right] d t, \\
& \begin{aligned}
\Sigma_{T}^{r, s,(2)}= & \frac{151 \theta}{140} \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d} \int_{0}^{T}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(C_{t}\right)\right)\left[\bar{C}_{t}^{g h, j k} \bar{C}_{t}^{a b, l m}\right. \\
& \left.\quad+\bar{C}_{t}^{a b, j k} \bar{C}_{t}^{g h, l m}\right] d t,
\end{aligned} \\
& \begin{aligned}
\Sigma_{T}^{r, s,(3)}=\frac{3}{2 \theta} \sum_{g, h, a, b=1}^{d} & \sum_{j, k, l, m=1}^{d} \int_{0}^{T}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(C_{t}\right)\right)\left[C_{t}(g h, j k) \bar{C}_{t}^{a b, l m}\right. \\
& \left.\quad+C_{t}(a b, l m) \bar{C}_{t}^{g h, j k}+C_{t}(g h, l m) \bar{C}_{t}^{a b, j k}+C_{t}(a b, j k) \bar{C}_{t}^{g h, l m}\right] d t
\end{aligned}
\end{aligned}
$$

with

$$
C_{t}(g h, j k)=C_{g j, t} C_{h k, t}+C_{g k, t} C_{h j, t} .
$$

The convergence in Theorem 1 is stable in law (denoted $L-s$, see for example Aldous and Eagleson (1978) and Jacod and Protter (2012)). The limit is mixed gaussian and the precision of the estimators depends on the paths of the spot covariance and the volatility of volatility process. The rate of convergence $\Delta_{n}^{-1 / 4}$ has been shown to be the optimal for volatility of volatility estimation (under the assumption of no volatility jumps).

The asymptotic variance of the estimators depends on the tuning parameter $\theta$ whose choice may be crucial for the reliability of the inference. We document the sensitivity of the inference theory to the choice of the parameter $\theta$ in a Monte Carlo experiment (see Section 6).

### 4.3 Estimation of the Asymptotic Covariance Matrix

To provide a consistent estimator for the element $\Sigma_{T}^{r, s}$ of the asymptotic covariance matrix in Theorem 1, we introduce the following quantities:

$$
\begin{aligned}
\widehat{\Omega}_{T}^{r, s,(1)}=\Delta_{n} & \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(\widehat{C}_{i \Delta_{n}}\right)\right) \\
& \times\left[\widetilde{C}_{i \Delta_{n}}(g h, j k) \widetilde{C}_{i \Delta_{n}}(a b, l m)+\widetilde{C}_{i \Delta_{n}}(a b, j k) \widetilde{C}_{i \Delta_{n}}(g h, l m)\right], \\
\widehat{\Omega}_{T}^{r, s,(2)}= & \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(\widehat{C}_{i \Delta_{n}}\right)\right)\left[\frac{1}{2} \widehat{\lambda}_{i}^{n, g h} \widehat{\lambda}_{i}^{n, j k} \widehat{\lambda}_{i+2 k_{n}}^{n, a b} \widehat{\lambda}_{i+2 k_{n}}^{n, l m}\right. \\
& \left.+\frac{1}{2} \widehat{\lambda}_{i}^{n, a b} \widehat{\lambda}_{i}^{n, l m} \widehat{\lambda}_{i+2 k_{n}}^{n, g h} \widehat{\lambda}_{i+2 k_{n}}^{n, j k}+\frac{1}{2} \widehat{\lambda}_{i}^{n, a b} \widehat{\lambda}_{i}^{n, j k} \widehat{\lambda}_{i+2 k_{n}}^{n, g h} \widehat{\lambda}_{i+2 k_{n}}^{n, l m}+\frac{1}{2} \widehat{\lambda}_{i}^{n, g h} \widehat{\lambda}_{i}^{n, l m} \widehat{\lambda}_{i+2 k_{n}}^{n, a b} \widehat{\lambda}_{\left.i+2 k_{n}\right]}^{n, j k}\right], \\
\widehat{\Omega}_{T}^{r, s,(3)}= & \frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(\widehat{C}_{i \Delta_{n}}\right)\right) \\
& {\left[\widetilde{C}_{i \Delta_{n}}(g h, j k) \hat{\lambda}_{i}^{n, a b} \widehat{\lambda}_{i}^{n, l m}+\widetilde{C}_{i \Delta_{n}}(a b, l m) \widehat{\lambda}_{i}^{n, g h} \widehat{\lambda}_{i}^{n, j k}\right.} \\
& +\widetilde{C}_{i \Delta_{n}}(g h, l m) \widehat{\lambda}_{i}^{n, a b} \widehat{\lambda}_{i}^{n, j k}+\left(\widetilde{C}_{i \Delta_{n}}(a b, j k) \widehat{\lambda}_{i}^{n, g h} \widehat{\lambda}_{i}^{n, l m}\right],
\end{aligned}
$$

with $\widehat{\lambda}_{i}^{n, j k}=\widehat{C}_{i+k_{n}}^{n, j k}-\widehat{C}_{i}^{n, j k}$ and $\widetilde{C}_{i \Delta_{n}}(g h, j k)=\left(\widehat{C}_{g j, i \Delta_{n}} \widehat{C}_{h k, i \Delta_{n}}+\widehat{C}_{g k, i \Delta_{n}} \widehat{C}_{h j, i \Delta_{n}}\right)$.
The following result holds,
Theorem 2. Suppose the assumptions of Theorem 1 hold. Then, as $\Delta_{n} \rightarrow 0$,

$$
\begin{align*}
& \frac{6}{\theta^{3}} \widehat{\Omega}_{T}^{r, s,(1)} \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(1)},  \tag{30}\\
& \frac{3}{2 \theta}\left[\widehat{\Omega}_{T}^{r, s,(3)}-\frac{6}{\theta} \widehat{\Omega}_{T}^{r, s,(1)}\right] \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(3)}, \text { and }  \tag{31}\\
& \frac{151 \theta}{140} \frac{9}{4 \theta^{2}}\left[\widehat{\Omega}_{T}^{r, s,(2)}+\frac{4}{\theta^{2}} \widehat{\Omega}_{T}^{r, s,(1)}-\frac{4}{3} \widehat{\Omega}_{T}^{r, s,(3)}\right] \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(2)} \tag{32}
\end{align*}
$$

The estimated matrix $\widehat{\Sigma}_{T}$ is symmetric but is not guaranteed to be positive semi-definite. By Theorem $1, \widehat{\Sigma}_{T}$ is positive semi-definite in large samples. An interesting question is the estimation of the asymptotic variance using subsampling or bootstrap methods, and we leave it for future research.

Remark 1: The rate of convergence in equation (30) can be shown to be $\Delta_{n}^{-1 / 2}$, and the rate of convergence in (31) and (32) can be shown to be $\Delta_{n}^{-1 / 4}$.

Remark 2: In the one-dimensional case ( $d=1$ ), much simpler estimators of $\Sigma_{T}^{r, s,(2)}$ can be constructed using the quantities $\widehat{\lambda}_{i}^{n, j k} \widehat{\lambda}_{i}^{n, l m} \widehat{\lambda}_{i+k_{n}}^{n, g h} \widehat{\lambda}_{i+k_{n}}^{n, x y}$ or $\hat{\lambda}_{i}^{n, j k} \widehat{\lambda}_{i}^{n, l m} \widehat{\lambda}_{i}^{n, g h} \widehat{\lambda}_{i}^{n, x y}$ as in Vetter (2015).

However, in the multidimensional case, the latter quantities do not identify separately the quantity ${\overline{C_{t}}}^{j k, l m}{\overline{C_{t}}}^{g h, x y}$ since the combination ${\overline{C_{t}}}^{j k, l m}{\overline{C_{t}}}^{g h, x y}+{\overline{C_{t}}}^{j k, g h}{\overline{C_{t}}}^{l m, x y}+{\overline{C_{t}}}^{j k, x y}{\overline{C_{t}}}^{g h, l m}$ shows up in a non-trivial way in the limit of the estimator.

Corollary 3. Let $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}\right.$ denote either $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}^{A N}\right.$ or $\left[H_{r}\left(\widehat{C), G_{r}}(C)\right]_{T}^{L I N}\right.$ defined in equations (16) and (18). Suppose the assumptions of Theorem 1 hold. Then, as $\Delta_{n} \rightarrow 0$,

$$
\Delta_{n}^{-1 / 4} \widehat{\Sigma}_{T}^{-1 / 2}\left(\begin{array}{c}
{\left[H_{1}\left(\widehat{C), G_{1}}(C)\right]_{T}-\left[H_{1}(C), G_{1}(C)\right]_{T}\right.}  \tag{33}\\
\vdots \\
{\left[H_{\kappa}\left(\widehat{C), G_{\kappa}}(C)\right]_{T}-\left[H_{\kappa}(C), G_{\kappa}(C)\right]_{T}\right.}
\end{array}\right) \xrightarrow{L} N\left(0, I_{\kappa}\right) .
$$

In the above, we use $L$ to denote the convergence in distribution and $I_{\kappa}$ the identity matrix of order $\kappa$. Corollary 3 states the standardized asymptotic distribution, which follows directly from the properties of the stable-in-law convergence. Similarly, by the Delta method, standardized asymptotic distribution can also be derived for the estimators of the quantities in (21). These standardized distributions allow the construction of confidence intervals for all the latent quantities of the form $\left[H_{r}(C), G_{r}(C)\right]_{T}$ and, more generally, functions of these quantities.

### 4.4 Tests

As an illustration of application of the general theory, we provide three tests about the dependence of Idiosyncratic Volatility. Our framework allows to test general hypotheses about the joint dynamics of any subset of the available stocks. The three examples below are stated for one pair of stocks, and correspond to the tests we implement in the empirical and Monte Carlo studies.

First, one can test for the absence of dependence between the IdioVols of the returns on assets $j$ and $s$,

$$
\begin{equation*}
H_{0}^{1}:\left[C_{Z j}, C_{Z s}\right]_{T}=0 \text { vs } H_{1}^{1}:\left[C_{Z j}, C_{Z s}\right]_{T} \neq 0 . \tag{34}
\end{equation*}
$$

The null hypothesis $H_{0}^{1}$ is rejected whenever the t-test exceeds the $\alpha / 2$-quantile of the standard normal distribution, $Z_{\alpha}$,

$$
\begin{equation*}
\Delta_{n}^{-1 / 4} \frac{\left|\left[\widehat{C Z j}, C_{Z s}\right]_{T}\right|}{\sqrt{\widehat{\operatorname{AVAR}}\left(C_{Z j}, C_{Z s}\right)}}>Z_{\alpha / 2} \tag{35}
\end{equation*}
$$

Second, we can test for all the IdioVol factors $\Pi$ being irrelevant to explain the dynamics of IdioVol shocks of stock $j$,

$$
\begin{equation*}
H_{0}^{2}:\left[C_{Z j}, \Pi\right]_{T}=0 \text { vs } H_{1}^{2}:\left[C_{Z j}, \Pi\right]_{T} \neq 0 . \tag{36}
\end{equation*}
$$

Under this null hypothesis, the vector of IdioVol factor loadings equals zero, $\gamma_{Z_{j}}=0$. The null hypothesis $H_{0}^{2}$ is rejected when

$$
\begin{equation*}
\Delta_{n}^{-1 / 4}\left(\left[\widehat{C_{Z j}, \Pi}\right]_{T}\right)^{\top}\left(\widehat{\operatorname{AVAR}}\left(C_{Z j}, \Pi\right)\right)^{-1}\left[\widehat{C_{Z j}, \Pi}\right]_{T}>\mathcal{X}_{d_{\Pi}, 1-\alpha}^{2}, \tag{37}
\end{equation*}
$$

where $d_{\Pi}$ denotes the number of IdioVol factors, and where $\mathcal{X}_{d_{q}, 1-\alpha}^{2}$ is the $(1-\alpha)$ quantile of the $\mathcal{X}_{d_{q}}^{2}$ distribution. One can of course also construct a t-test for irrelevance of any one particular IdioVol factor. The final example is a test for absence of dependence between the residual IdioVols,

$$
\begin{equation*}
H_{0}^{3}:\left[C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T}=0 \text { vs } H_{1}^{3}:\left[C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T} \neq 0 . \tag{38}
\end{equation*}
$$

The null can be rejected when the following t-test exceeds the critical value,

$$
\begin{equation*}
\Delta_{n}^{-1 / 4} \frac{\left|\left[C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right]_{T}\right|}{\sqrt{\widehat{\operatorname{AVAR}}\left(C_{Z j}^{\text {resid }}, C_{Z s}^{\text {resid }}\right)}}>Z_{\alpha / 2} \tag{39}
\end{equation*}
$$

Each of the above estimators

$$
\left[\widehat{C_{Z j}, C_{Z s}}\right]_{T},\left[\widehat{C_{Z j}, \Pi}\right]_{T}, \text { and }\left[C_{Z j}^{r e s i d}, C_{Z s}^{r e s i d}\right]_{T}
$$

can be obtained by choosing appropriate pair(s) of transformations $H$ and $G$ in the general estimator $\left[H(\widehat{C), G}(C)]_{T}\right.$, see Section 3 for details. Any of the two types of the latter estimator can be used,

$$
\left[H ( \widehat { C ) , G } ( C ) ] _ { T } ^ { A N } \quad \text { or } \left[H(\widehat{C), G}(C)]_{T}^{L I N}\right.\right.
$$

For the first two tests, the expression for the true asymptotic variance, AVAR, is obtained using Theorem 1 and its estimation follows from Theorem 2. The asymptotic variance in the third test is obtained by applying the Delta method to the joint convergence result in Theorem 1. The expression for the estimator of the asymptotic variance, $\widehat{\text { AVAR, follows from Theorem 2. Under R-FM and }}$ the assumptions of Theorem 1, Corollary 3 implies that the asymptotic size of the two types of
tests for the null hypotheses $H_{0}^{1}$ and $H_{0}^{2}$ is $\alpha$, and their power approaches 1 . The same properties apply for the tests of the null hypotheses $H_{0}^{3}$ with our R-FM and IdioVol-FM representations.

Theoretically, it is possible to test for absence of dependence in the IdioVols at each point in time. In this case the null hypothesis is $H_{0}^{1 \prime}:\left[C_{Z j}, C_{Z s}\right]_{t}=0$ for all $0 \leq t \leq T$, which is, in theory, stronger than our $H_{0}^{1 \prime}$. In particular, Theorem 1 can be used to set up KolmogorovSmirnov type of tests for $H_{0}^{\prime 1}$ in the same spirit as Vetter (2015). However, we do not pursue this direction in the current paper for two reasons. First, the testing procedure would be more involved. Second, empirical evidence suggests nonnegative dependence between IdioVols, which means that in practice, it is not too restrictive to assume $\left[C_{Z j}, C_{Z s}\right]_{t} \geq 0 \forall t$, under which $H_{0}^{1}$ and $H_{0}^{1 \prime}$ are equivalent.

## 5 Empirical Analysis

We apply our methods to study the cross-sectional dependence in IdioVols using high frequency data. One of our main findings is that stocks' IdioVols co-move strongly with the market volatility. This is a quite surprising finding. It is of course well known that the total volatility of stocks moves with the market volatility. However, we stress that we find that the strong effect is still present when considering the IdioVols.

We use full record transaction prices from NYSE TAQ database for 30 constituents of the DJIA index over the time period 2003-2012, see Table 1. After removing the non-trading days, our sample contains 2517 days. The selected stocks were the constituents of the DJIA index in 2007. We also use the high-frequency data on nine industry Exchange-Traded Funds, ETFs (Consumer Discretionary, Consumer Staples, Energy, Financial, Health Care, Industrial, Materials, Technology, and Utilities), and the high-frequency size and value Fama-French factors, see Aït-Sahalia, Kalnina, and Xiu (2020). For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. until 4 p.m. We clean the data following the procedure suggested by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), remove the overnight returns and then sample at 5 minutes. This sparse sampling has been widely used in the literature because the effect of the microstructure noise and potential asynchronicity of the data is less important at this frequency, see also Liu, Patton, and Sheppard (2015). The jump truncation threshold is the same as in simulations, see Section 6. The number of observations in the local window is taken as in Theorem 1 to be $k_{n}=\theta \Delta_{n}^{-1 / 2}$. We take $\theta=2.5$ and $\Delta_{n}=1 / 252 /(6.5 \times 12)$, i.e., $\Delta_{n}$ is 5 minutes
(with one year being a unit of time), which corresponds to the local window of approximately one week.

To obtain the Idiosyncratic Volatilities, the preliminary step is to estimate the Return Factor Model (R-FM) for each stock. Figures F. 1 and F. 2 contain plots of the time series of the estimated $R_{Y j}^{2}$ of the R-FM for each stock. ${ }^{8}$ Each plot contains monthly $R_{Y j}^{2}$ from two Return Factor Models, CAPM and the Fama-French regression with market, size, and value factors. Figures F. 1 and F. 2 show that these time series of all stocks follow approximately the same trend with a considerable increase in the contribution around the crisis year 2008. Higher $R_{Y j}^{2}$ indicates that the systematic risk is relatively more important, which is typical during crises. $R_{Y j}^{2}$ is consistently higher in the Fama-French regression model compared to the CAPM regression model, albeit not by much. We proceed to investigate the dynamic properties of the panel of Idiosyncratic Volatilities.

We first investigate the dependence in the (total) Idiosyncratic Volatilities. Our panel has 435 pairs of stocks. For each pair of stocks, we compute the correlation between the IdioVols, $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$, see Section 3.2 for the implementation details. All pairwise correlations are positive in our sample, and their average is 0.55 . We find that both types of estimators, AN and LIN, produce very similar results and report only the AN estimator for brevity. Figure 1 maps the network of dependency in the IdioVol. We simultaneously test 435 hypotheses of no correlation, and Figure 1 connects only the assets, for which the null is rejected. We account for multiple testing by controlling the false discovery rate at $5 \%$. Overall, Figure 1 shows that the cross-sectional dependence between the IdioVols is very strong.

Could missing factors in the R-FM provide an explanation? Omitted return factors in the R-FM are captured by the idiosyncratic returns, and can therefore induce correlation between the estimated IdioVols, provided these missing return factors have non-negligible volatility of volatility. To investigate this possibility, we consider the correlations between idiosyncratic returns, $\operatorname{Corr}\left(Z_{i}, Z_{j}\right) .{ }^{9}$ Table 2 presents a summary of how estimates $\operatorname{Corr}\left(Z_{i}, Z_{j}\right)$ are related to the estimates of correlation in IdioVols, $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$. In particular, different rows in Table 2 dis-

[^7]

Figure 1: The network of dependencies in total IdioVols. The color and thickness of each line is proportional to the estimated value of $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$, the quadratic-covariation based correlation between the IdioVols, defined in equation (8) (red and thick lines indicate high correlation). We simultaneously test 435 null hypotheses of no correlation, and the lines are only plotted when the null is rejected.
play average values of $\widehat{\operatorname{Corr}}\left(C_{Z i}, C_{Z j}\right)$ among those pairs, for which $\left|\widehat{\operatorname{Corr}}\left(Z_{i}, Z_{j}\right)\right|$ is below some threshold. For example, the last-but-one row in Table 2 indicates that there are 56 pairs of stocks with $\left|\widehat{\operatorname{Corr}}\left(Z_{i}, Z_{j}\right)\right|<0.01$, and among those stocks, the average correlation between IdioVols, $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$, is estimated to be 0.579 . We observe that $\widehat{\operatorname{Corr}}\left(C_{Z i}, C_{Z j}\right)$ is virtually the same compared to pairs of stocks with high $\left|\operatorname{Corr}\left(Z_{i}, Z_{j}\right)\right|$. These results suggest that missing return factors cannot explain dependence in IdioVols for all considered stocks. This finding is in line with the empirical analysis of Herskovic, Kelly, Lustig, and Nieuwerburgh (2016) with daily and monthly returns.

To understand the source of the strong cross-sectional dependence in the IdioVols, we consider the Idiosyncratic Volatility Factor Model (IdioVol-FM) of Section 2. We first use the market volatility as the only IdioVol factor. ${ }^{10}$ Table 3 reports the estimates of the IdioVol loading ( $\widehat{\gamma}_{Z i}$ ) and the $R^{2}$ of the IdioVol-FM $\left(R_{Z i}^{2, I d i o V o l-F M}\right.$, see equation (10)). Table 3 uses two different definitions of IdioVol, one defined with respect to CAPM, and a second IdioVol defined with respect to Fama-French three factor model. For every stock, the estimated IdioVol factor loading is positive, suggesting that the Idiosyncratic Volatility co-moves with the market volatility. Next, Figure

[^8]2 shows the implications for the cross-section of the one-factor IdioVol-FM when the IdioVol is defined with respect to CAPM. The average pairwise correlations between the residual IdioVols, $\widehat{\operatorname{Corr}}\left(C_{Z i}, C_{Z j}\right)$, decrease to 0.25 . However, the market volatility cannot explain all cross-sectional dependence in residual IdioVols, as evidenced by the remaining links in Figure 2.

Finally, we consider an IdioVol-FM with ten IdioVol factors, market volatility and the volatilities of nine industry ETFs. Figure 3 shows the implications for the cross-section of this ten-factor IdioVol-FM when the IdioVol is defined with respect to CAPM. The average pairwise correlations between the residual IdioVols, $\widehat{\operatorname{Corr}}\left(C_{Z i}, C_{Z j}\right)$, decrease further to 0.18 . As we can see from Figure 3 , the remaining cross-sectional dependence is statistically insignificant after accounting for multiple testing. ${ }^{11}$ For completeness, Figure F. 3 in the Appendix graphs all correlations of Figures $1-3$.

For comparison, we also calculate the naive estimators, see equation (15). Of course, we do not have valid confidence intervals to accompany these estimators. In our data set, the relative differences between the naive and the bias-corrected estimators are around $4 \%$ for $\gamma_{Z_{j}}$, they range, across stocks, between 3 and $6 \%$ for $R_{Z j}^{2, \text { IdioVol-FM }}$, between 2 and $7 \%$ for $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$, and between $-42 \%$ and $7 \%$ for $\operatorname{Corr}\left(C_{Z i}^{\text {resid }}, C_{Z j}^{r e s i d}\right)$. We find that in the instances where the differences are small, the multiplicative bias, i.e., the factor $2 / 3$, dominates the additive bias both in the numerator and the denominator, so that the multiplicative bias approximately cancels out. We find that the differences between the bias-corrected and naive estimators increase if we only consider the time period before or after the financial crisis of 2009.

[^9]

Figure 2: The network of dependencies in residual IdioVols with a single IdioVol factor: the market variance.


Figure 3: The network of dependencies in residual IdioVols with ten IdioVol factors: the market variance and the variances of nine industry ETFs.

In both figures, the color and thickness of each line is proportional to the estimated value of $\operatorname{Corr}\left(C_{Z i}^{r e s i d}, C_{Z j}^{r e s i d}\right)$, the quadratic-covariation based correlation between the residual IdioVols, defined in equation (9), of each pair of stocks (red and thick lines indicate high correlation). We simultaneously test 435 null hypotheses of no correlation, and the lines are only plotted when the null is rejected.

| Sector | Stock | Ticker |
| :--- | :--- | :--- |
| Financial | American International Group, Inc. | AIG |
|  | American Express Company | AXP |
|  | Citigroup Inc. | C |
|  | JPMorgan Chase \& Co. | JPM |
| Energy | Chevron Corp. | CVX |
|  | Exxon Mobil Corp. | XOM |
| Consumer Staples | Coca Cola Company | KO |
|  | Altria | MO |
|  | The Procter \& Gamble Company | PG |
|  | Wal-Mart Stores | WMT |
| Industrials | Boeing Company | BA |
|  | Caterpillar Inc. | CAT |
|  | General Electric Company | GE |
|  | Honeywell International Inc | HON |
|  | 3M Company | MMM |
|  | United Technologies | UTX |
| Technology | Hewlett-Packard Company | HPQ |
|  | International Bus. Machines | IBM |
|  | Intel Corp. | INTC |
|  | Microsoft Corporation | MSFT |
| Health Care | Johnson \& Johnson | JNJ |
|  | Merck \& Co. | MRK |
|  | Pfizer Inc. | PFE |
| Consumer Discretionary | The Walt Disney Company | DIS |
|  | Home Depot Inc | HD |
|  | McDonald's Corporation | MCD |
| Materials | Alcoa Inc. | AA |
|  | E.I. du Pont de Nemours \& Company | DD |
| Telecommunications Services | AT\&T Inc. | T |
|  | Verizon Communications Inc. | VZ |

Table 1: The table lists the stocks used in the empirical application (for the time period 2003-2012). They are the 30 constituents of DJIA in 2007. The first column provides the Global Industry Classification Standard (GICS) sectors, the second the names of the companies and the third their tickers.

| $\widehat{\widehat{\operatorname{Corr}}\left(Z_{i}, Z_{j}\right) \mid}$ | CAPM |  |  | FF3 Model |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Pairs | Avg $\left\|\widehat{\operatorname{Corr}}\left(Z_{i}, Z_{j}\right)\right\|$ | Avg $\widehat{C o r r}\left(C_{Z i}, C_{Z j}\right)$ | Pairs | $\operatorname{Avg}\left\|\widehat{\operatorname{Corr}}\left(Z_{i}, Z_{j}\right)\right\|$ | $\operatorname{Avg} \widehat{\operatorname{Corr}}\left(C_{Z i}, C_{Z j}\right)$ |
| $<0.6$ | 435 | 0.038 | 0.510 | 435 | 0.038 | 0.512 |
| $<0.5$ | 434 | 0.036 | 0.509 | 434 | 0.037 | 0.512 |
| <0.4 | 434 | 0.036 | 0.509 | 434 | 0.037 | 0.512 |
| <0.3 | 434 | 0.036 | 0.509 | 434 | 0.037 | 0.512 |
| <0.2 | 431 | 0.035 | 0.508 | 430 | 0.035 | 0.511 |
| $<0.1$ | 403 | 0.028 | 0.503 | 404 | 0.029 | 0.506 |
| < 0.075 | 383 | 0.025 | 0.500 | 382 | 0.026 | 0.502 |
| < 0.050 | 315 | 0.018 | 0.487 | 316 | 0.019 | 0.489 |
| < 0.025 | 177 | 0.006 | 0.447 | 178 | 0.007 | 0.452 |
| < 0.010 | 80 | 0.001 | 0.415 | 81 | 0.002 | 0.414 |
| < 0.005 | 43 | 0.000 | 0.385 | 41 | 0.001 | 0.409 |

Table 2: Each row in this table describes the subset of pairs of stocks with $\left|\operatorname{Corr} \widehat{\left(Z_{i}, Z_{j}\right)}\right|$ below a threshold in column one. The table considers two R-FMs: the left panel defines the IdioVol with respect to CAPM, and the right panel defines the IdioVol with respect to the three-factor Fama-French model. In both cases, the market volatility is the only IdioVol factor. Each panel reports three quantities for the given subset of pairs: the number of pairs, average absolute pairwise correlation in idiosyncratic returns, and average pairwise correlation between IdioVols.

|  | CAPM |  |  |  | FF3 Model <br> Stock |  |  | $\widehat{\gamma}_{z}$ | $\widehat{R}_{Z}^{2, \text { IdioVol-FM }}$ | p-val | $\widehat{\gamma}_{z}$ | $\widehat{R}_{Z}^{, \text {IdioVol-FM }}$ | p-val |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| AIG | 1.49 | 0.02 | 0.093 | 1.53 | 0.02 | 0.085 |  |  |  |  |  |  |  |
| AXP | 3.02 | 0.27 | 0.146 | 2.98 | 0.27 | 0.149 |  |  |  |  |  |  |  |
| C | 3.46 | 0.108 | 0.007 | 3.48 | 0.11 | 0.007 |  |  |  |  |  |  |  |
| JPM | 2.44 | 0.20 | 0.007 | 2.46 | 0.21 | 0.006 |  |  |  |  |  |  |  |
| CVX | 1.08 | 0.51 | 0.030 | 1.07 | 0.51 | 0.030 |  |  |  |  |  |  |  |
| XOM | 0.60 | 0.48 | 0.044 | 0.61 | 0.49 | 0.043 |  |  |  |  |  |  |  |
| KO | 0.33 | 0.58 | 0.012 | 0.33 | 0.58 | 0.011 |  |  |  |  |  |  |  |
| MO | 0.44 | 0.35 | 0.001 | 0.44 | 0.35 | 0.001 |  |  |  |  |  |  |  |
| PG | 0.43 | 0.63 | 0.001 | 0.43 | 0.63 | 0.002 |  |  |  |  |  |  |  |
| WMT | 0.45 | 0.58 | 0.006 | 0.45 | 0.56 | 0.008 |  |  |  |  |  |  |  |
| BA | 0.47 | 0.42 | 0.003 | 0.48 | 0.44 | 0.003 |  |  |  |  |  |  |  |
| CAT | 0.69 | 0.49 | 0.009 | 0.69 | 0.48 | 0.009 |  |  |  |  |  |  |  |
| GE | 1.14 | 0.26 | 0.003 | 1.15 | 0.26 | 0.002 |  |  |  |  |  |  |  |
| HON | 0.53 | 0.44 | 0.014 | 0.53 | 0.43 | 0.014 |  |  |  |  |  |  |  |
| MMM | 0.39 | 0.55 | 0.000 | 0.38 | 0.54 | 0.000 |  |  |  |  |  |  |  |
| UTX | 0.50 | 0.52 | 0.003 | 0.50 | 0.53 | 0.004 |  |  |  |  |  |  |  |
| HPQ | 0.65 | 0.33 | 0.004 | 0.66 | 0.34 | 0.004 |  |  |  |  |  |  |  |
| IBM | 0.35 | 0.48 | 0.011 | 0.35 | 0.47 | 0.012 |  |  |  |  |  |  |  |
| INTC | 0.46 | 0.46 | 0.003 | 0.46 | 0.46 | 0.003 |  |  |  |  |  |  |  |
| MSFT | 0.68 | 0.52 | 0.008 | 0.67 | 0.51 | 0.010 |  |  |  |  |  |  |  |
| JNJ | 0.41 | 0.68 | 0.007 | 0.40 | 0.67 | 0.007 |  |  |  |  |  |  |  |
| MRK | 0.54 | 0.32 | 0.001 | 0.54 | 0.32 | 0.001 |  |  |  |  |  |  |  |
| PFE | 0.43 | 0.34 | 0.002 | 0.43 | 0.34 | 0.001 |  |  |  |  |  |  |  |
| DIS | 0.57 | 0.48 | 0.001 | 0.58 | 0.49 | 0.001 |  |  |  |  |  |  |  |
| HD | 0.66 | 0.45 | 0.010 | 0.66 | 0.45 | 0.010 |  |  |  |  |  |  |  |
| MCD | 0.29 | 0.29 | 0.003 | 0.29 | 0.29 | 0.003 |  |  |  |  |  |  |  |
| AA | 3.03 | 0.41 | 0.019 | 3.04 | 0.42 | 0.018 |  |  |  |  |  |  |  |
| DD | 0.61 | 0.59 | 0.001 | 0.61 | 0.59 | 0.001 |  |  |  |  |  |  |  |
| T | 0.76 | 0.45 | 0.003 | 0.76 | 0.44 | 0.003 |  |  |  |  |  |  |  |
| VZ | 0.54 | 0.55 | 0.000 | 0.54 | 0.54 | 0.001 |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |

Table 3: Estimates of the IdioVol factor loading $\left(\hat{\gamma}_{Z}\right.$, see equation (6)), and the contribution of the market volatility to the variation in the IdioVols ( $\widehat{R}_{Z}^{2, I d i o V o l-F M}$, see equation (10)). The table considers two R-FMs: the left panel defines the IdioVol with respect to CAPM, and the right panel defines the IdioVol with respect to the three-factor Fama-French model. In both cases, the market volatility is the only IdioVol factor. P-val is the p-value of the test of the absence of dependence between the IdioVol and the market volatility for a given individual stock, see equation (37).

## 6 Monte Carlo

This section investigates the finite sample properties of our estimators and tests. The data generating process (DGP) is similar to that of Li, Todorov, and Tauchen (2013) and is constructed as follows. Denote by $Y_{1}$ and $Y_{2}$ the log-prices of two individual stocks, and by $X$ the log-price of the market portfolio. Recall that the superscript $c$ indicates the continuous part of a process. We assume

$$
d X_{t}=d X_{t}^{c}+d J_{3, t}, \quad d X_{t}^{c}=\sqrt{C_{X, t}} d W_{t},
$$

and, for $j=1,2$,

$$
d Y_{j, t}=\beta_{t} d X_{t}^{c}+d \widetilde{Y}_{j, t}^{c}+d J_{j, t}, \quad d \widetilde{Y}_{j, t}^{c}=\sqrt{C_{Z j, t}} d \widetilde{W}_{j, t} .
$$

In the above, $C_{X}$ is the spot volatility of the market portfolio, $\widetilde{W}_{1}$ and $\widetilde{W}_{2}$ are Brownian motions with $\operatorname{Corr}\left(d \widetilde{W}_{1, t}, d \widetilde{W}_{2, t}\right)=0.4$, and $W$ is an independent Brownian motion; $J_{1}, J_{2}$, and $J_{3}$ are independent compound Poisson processes with intensity equal to 2 jumps per year and jump size distribution $N\left(0,0.02^{2}\right)$. The beta process is time-varying and is specified as $\beta_{t}=0.5+0.1 \sin (100 t)$.

We next specify the volatility processes. As our building blocks, we first generate four processes $f_{1}, \ldots, f_{4}$ as mutually independent Cox-Ingersoll-Ross processes,

$$
\begin{aligned}
& d f_{1, t}=5\left(0.09-f_{1, t}\right) d t+0.35 \sqrt{f_{1, t}}\left(-0.8 d W_{t}+\sqrt{1-0.8^{2}} d B_{1, t}\right), \\
& d f_{j, t}=5\left(0.09-f_{j, t}\right) d t+0.35 \sqrt{f_{j, t}} d B_{j, t}, \text { for } j=2,3,4,
\end{aligned}
$$

where $B_{1}, \ldots, B_{4}$ are independent standard Brownian Motions, which are also independent from the Brownian Motions of the return Factor Model. ${ }^{12}$ We use the first process $f_{1}$ as the market volatility, i.e., $C_{X, t}=f_{1, t}$. We use the other three processes $f_{2}, f_{3}$, and $f_{4}$ to construct three different specifications for the IdioVol processes $C_{Z 1, t}$ and $C_{Z 2, t}$, see Table 4 for details. The common Brownian Motion $W_{t}$ in the market portfolio price process $X_{t}$ and its volatility process $C_{X, t}=f_{1, t}$ generates a leverage effect for the market portfolio. The value of the leverage effect is -0.8 , which is standard in the literature, see Kalnina and Xiu (2017), Aït-Sahalia, Fan, and Li (2013) and Aït-Sahalia, Fan, Laeven, Wang, and Yang (2017). ${ }^{13}$

[^10]|  | $C_{Z 1, t}$ | $C_{Z 2, t}$ |
| :--- | :---: | :---: |
| Model 1 | $0.1+1.5 f_{2, t}$ | $0.1+1.5 f_{3, t}$ |
| Model 2 | $0.1+0.6 C_{X, t}+0.4 f_{2, t}$ | $0.1+0.5 C_{X, t}+0.5 f_{3, t}$ |
| Model 3 | $0.1+0.45 C_{X, t}+f_{2, t}+0.4 f_{4, t}$ | $0.1+0.35 C_{X, t}+0.3 f_{3, t}+0.6 f_{4, t}$ |

Table 4: Different specifications for the Idiosyncratic Volatility processes $C_{Z 1, t}$ and $C_{Z 2, t}$.

We set the time span $T$ equal to 1,260 or 2,520 days, which correspond approximately to 5 and 10 business years. These values are standard in the nonparametric leverage effect estimation literature (see Aït-Sahalia, Fan, and Li (2013) and Kalnina and Xiu (2017)), where the rate of convergence is also $\Delta^{-1 / 4}$. Each day consists of 6.5 trading hours. We consider two different values for the sampling frequency, $\Delta_{n}=1$ minute and $\Delta_{n}=5$ minutes. We follow Li, Todorov, and Tauchen (2016) and set the jump truncation threshold $u_{n}$ in day $t$ at $3 \widehat{\sigma}_{t} \Delta_{n}^{0.49}$, where $\widehat{\sigma}_{t}$ is the squared root of the annualized bipower variation of Barndorff-Nielsen and Shephard (2004). We choose four different values for the width of the subsamples, which corresponds to $\theta=1.5,2,2.5$ and 3 (recall that the number of observations in a window is $k_{n}=\theta / \sqrt{\Delta_{n}}$ ). We use 10,000 Monte Carlo replications in all the experiments.

We first investigate the finite sample properties of the estimators (using Model 3). We consider the following estimands:

- the IdioVol factor loading of the first stock, $\gamma_{Z 1}$,
- the contribution of the market volatility to the variation of the IdioVol of the first stock $R_{Z 1}^{2, \text { Idio Vol-FM }}$,
- the correlation between the Idiosyncratic Volatilities of stocks 1 and $2, \operatorname{Corr}\left(C_{Z 1}, C_{Z 2}\right)$,
- the correlation between the residual Idiosyncratic Volatilities, $\operatorname{Corr}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$.

The interpretation of simulation results is simpler when the quantities of interest do not change across simulations. To achieve that, we generate once and keep fixed the paths of the processes $C_{X, t}$ and $\left(f_{j, t}\right)_{0 \leq j \leq 4}$ and replicate several times the other parts of the DGP.

In Table 5, we report the median bias, the interquartile range (IQR), and the RMSE of the two type of the bias-corrected estimators as well as the naive estimator for each estimand using 5 minutes data over 10 years. Consider first the comparison of the AN and LIN estimators. One $\left(X_{t}, Y_{1, t}, Y_{2, t}\right)^{\prime}$ and equation (2) for the volatility matrix of this vector.
does not consistently overperform the other in terms of the bias or the IQR. Interestingly, in terms of the RMSE, the LIN estimator outperforms the AN estimator in every scenario considered. The naive estimators are substantially biased. The comparison of the bias-corrected estimators and the naive estimators reveals the usual bias-variance trade-off, as the bias-corrected estimators have smaller bias but larger IQR than the naive estimator. In terms of RMSE, the bias-corrected estimators generally outperform the naive estimator: RMSE is significantly lower when estimating $\gamma_{Z 1}, R_{Z 1}^{2, \text { IdioVol-FM }}$, or $\operatorname{Corr}\left(C_{Z 1}, C_{Z 2}\right)$, while the results for $\operatorname{Corr}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ are mixed.

It is also informative to see how these results change when we increase the sampling frequency. In Table 6, we report the results with $\Delta_{n}=1$ minute in the same setting. The qualitative conclusions of Table 5 remain true in Table 6. Compared to Table 5, the bias and IQR are smaller. However, the magnitude of the decrease of the IQR is small.

Finally, Table 7 contains results from same experiment using data sampled at one minute over 5 years. Despite using more than twice as many observations than in the first experiment, the precision is not as good. In other words, increasing the time span is more effective for precision gain than increasing the sampling frequency. The qualitative conclusions generally remain the same as in Table 5.

Next, we study the empirical rejection probabilities of the three statistical tests as outlined in Section 4.4. The first null hypothesis is the absence of dependence between the IdioVols, $H_{0}^{1}$ : $\left[C_{Z 1}, C_{Z 2}\right]_{T}=0$. The second null hypothesis we test is the absence of dependence between the IdioVol of the first stock and the market volatility, $H_{0}^{2}:\left[C_{Z 1}, C_{X}\right]_{T}=0$. The third null hypothesis is the absence of dependence in the two residual IdioVols, $H_{0}^{3}:\left[C_{Z 1}^{r e s i d}, C_{Z 2}^{\text {resid }}\right]_{T}=0$. We use Model 1 for the first two hypotheses and Model 2 for the third hypothesis.

The three panels of Table 8 contain the empirical rejection probabilities for the three null hypotheses. We present the results for two sampling frequencies ( $\Delta_{n}=1$ minute and $\Delta_{n}=$ 5 minutes) and the two type of estimators (AN and LIN). We see that the empirical rejection probabilities are reasonably close to the nominal size of the test. Neither type of estimator (AN or LIN) seems to dominate the other. Consistent with the asymptotic theory, the empirical rejection probabilities of the three tests become closer to the nominal size of the test when frequency is higher.

| $\widehat{\theta}$ | LIN |  |  |  | AN |  |  |  | Naive |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 |
|  | Median Bias |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | -0.007 | -0.004 | -0.005 | -0.011 | -0.032 | -0.027 | -0.025 | -0.028 | -0.257 | -0.230 | -0.209 | -0.177 |
| $\widehat{R}_{Z 1}^{2, I d i o V o l-F M}$ | -0.153 | -0.138 | -0.127 | -0.115 | -0.146 | -0.132 | -0.121 | -0.110 | -0.484 | -0.465 | -0.448 | -0.417 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | -0.129 | -0.104 | -0.086 | -0.059 | -0.147 | -0.118 | -0.100 | -0.070 | -0.342 | -0.334 | -0.325 | -0.307 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{r e s i d}, C_{Z 2}^{r e s i d}\right)$ | -0.089 | -0.064 | -0.045 | -0.018 | -0.109 | -0.082 | -0.061 | -0.029 | -0.245 | -0.239 | -0.232 | -0.218 |
|  | IQR |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.173 | 0.157 | 0.141 | 0.118 | 0.173 | 0.154 | 0.140 | 0.118 | 0.079 | 0.078 | 0.078 | 0.076 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | 0.180 | 0.166 | 0.154 | 0.133 | 0.201 | 0.185 | 0.170 | 0.141 | 0.040 | 0.042 | 0.044 | 0.046 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.279 | 0.257 | 0.238 | 0.211 | 0.321 | 0.289 | 0.266 | 0.229 | 0.039 | 0.041 | 0.043 | 0.048 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{r e s i d}\right)$ | 0.330 | 0.304 | 0.280 | 0.249 | 0.381 | 0.344 | 0.311 | 0.273 | 0.040 | 0.042 | 0.044 | 0.049 |
|  | RMSE |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.130 | 0.116 | 0.105 | 0.090 | 0.132 | 0.118 | 0.108 | 0.093 | 0.263 | 0.238 | 0.217 | 0.185 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | 0.206 | 0.185 | 0.170 | 0.150 | 0.242 | 0.192 | 0.174 | 0.152 | 0.484 | 0.466 | 0.449 | 0.418 |
| $\widehat{\operatorname{Corrr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.257 | 0.226 | 0.203 | 0.169 | 0.309 | 0.260 | 0.229 | 0.187 | 0.343 | 0.335 | 0.327 | 0.309 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | 0.300 | 0.261 | 0.235 | 0.199 | 0.394 | 0.309 | 0.266 | 0.213 | 0.247 | 0.241 | 0.234 | 0.221 |

[^11] $0.342, \operatorname{Corr}\left(C_{Z 1}, C_{Z 2}\right)=0.523, \operatorname{Corr}\left(C_{Z 1}^{r e s i d}, C_{Z 2}^{r e s i d}\right)=0.424$. Model 3 .

| $\theta$ | LIN |  |  |  | AN |  |  |  | Naive |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 |
|  | Median Bias |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | -0.034 | -0.029 | -0.022 | -0.013 | -0.052 | -0.044 | -0.036 | -0.025 | -0.304 | -0.295 | -0.275 | -0.267 |
| $\widehat{R}_{Z 1}^{2, I d i o V o l-F M}$ | -0.140 | -0.123 | -0.109 | -0.086 | -0.135 | -0.117 | -0.103 | -0.080 | -0.496 | -0.492 | -0.477 | -0.473 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | -0.146 | -0.128 | -0.114 | -0.091 | -0.163 | -0.143 | -0.129 | -0.104 | -0.327 | -0.323 | -0.321 | -0.317 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | -0.118 | -0.105 | -0.095 | -0.076 | -0.138 | -0.123 | -0.111 | -0.092 | -0.220 | -0.216 | -0.216 | -0.212 |
|  | IQR |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.147 | 0.132 | 0.118 | 0.100 | 0.146 | 0.131 | 0.117 | 0.099 | 0.062 | 0.062 | 0.063 | 0.063 |
| $\widehat{R}_{Z 1}^{2, I d i o}$ Vol-FM | 0.165 | 0.148 | 0.137 | 0.119 | 0.176 | 0.158 | 0.145 | 0.125 | 0.032 | 0.032 | 0.034 | 0.034 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.260 | 0.232 | 0.209 | 0.175 | 0.287 | 0.249 | 0.224 | 0.188 | 0.032 | 0.032 | 0.033 | 0.033 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | 0.312 | 0.280 | 0.254 | 0.211 | 0.341 | 0.303 | 0.273 | 0.225 | 0.032 | 0.032 | 0.033 | 0.033 |
|  | RMSE |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.115 | 0.102 | 0.091 | 0.076 | 0.121 | 0.106 | 0.095 | 0.078 | 0.307 | 0.299 | 0.279 | 0.271 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | 0.192 | 0.165 | 0.147 | 0.121 | 0.198 | 0.168 | 0.148 | 0.121 | 0.496 | 0.493 | 0.478 | 0.474 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.251 | 0.220 | 0.196 | 0.162 | 0.283 | 0.243 | 0.215 | 0.177 | 0.328 | 0.324 | 0.322 | 0.318 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | 0.291 | 0.249 | 0.221 | 0.182 | 0.760 | 0.279 | 0.245 | 0.199 | 0.221 | 0.218 | 0.218 | 0.214 |

[^12]| $\widehat{\theta}$ | LIN |  |  |  | AN |  |  |  | Naive |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 | 1.5 | 2 | 2.5 | 3 |
|  | Median Bias |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | -0.075 | -0.072 | -0.068 | -0.061 | -0.096 | -0.089 | -0.083 | -0.075 | -0.323 | -0.315 | -0.299 | -0.291 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | -0.183 | -0.169 | -0.155 | -0.139 | -0.183 | -0.169 | -0.156 | -0.137 | -0.500 | -0.496 | -0.484 | -0.480 |
| $\widehat{\text { Corr }}\left(C_{Z 1}, C_{Z 2}\right)$ | -0.187 | -0.169 | -0.161 | -0.145 | -0.214 | -0.194 | -0.185 | -0.166 | -0.321 | -0.316 | -0.317 | -0.313 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | -0.144 | -0.128 | -0.125 | -0.116 | -0.167 | -0.155 | -0.146 | -0.139 | -0.209 | -0.205 | -0.207 | -0.202 |
|  | IQR |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.229 | 0.205 | 0.184 | 0.154 | 0.225 | 0.202 | 0.184 | 0.154 | 0.092 | 0.092 | 0.093 | 0.093 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | 0.246 | 0.223 | 0.206 | 0.177 | 0.265 | 0.238 | 0.218 | 0.187 | 0.047 | 0.047 | 0.049 | 0.049 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.407 | 0.357 | 0.325 | 0.281 | 0.453 | 0.394 | 0.354 | 0.299 | 0.047 | 0.046 | 0.049 | 0.048 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | 0.475 | 0.419 | 0.387 | 0.324 | 0.529 | 0.462 | 0.420 | 0.352 | 0.047 | 0.047 | 0.049 | 0.049 |
|  | RMSE |  |  |  |  |  |  |  |  |  |  |  |
| $\widehat{\gamma}_{Z 1}$ | 0.184 | 0.165 | 0.150 | 0.127 | 0.192 | 0.172 | 0.156 | 0.134 | 0.330 | 0.321 | 0.307 | 0.298 |
| $\widehat{R}_{Z 1}^{2, \text { IdioVol-FM }}$ | 0.330 | 0.240 | 0.218 | 0.188 | 0.420 | 0.246 | 0.225 | 0.192 | 0.501 | 0.497 | 0.486 | 0.482 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}, C_{Z 2}\right)$ | 0.409 | 0.342 | 0.307 | 0.260 | 0.500 | 0.388 | 0.345 | 0.285 | 0.322 | 0.318 | 0.319 | 0.314 |
| $\widehat{\operatorname{Corr}}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)$ | 0.510 | 0.399 | 0.355 | 0.287 | 0.813 | 0.481 | 0.417 | 0.323 | 0.212 | 0.207 | 0.209 | 0.205 |

Table 7: Finite sample properties of our estimators using 5 years of data sampled at 1 minute. The true values are $\gamma_{Z 1}=0.450, R_{Z 1}^{2, \text { Idio Vol-FM }}=0.35$, $\operatorname{Corr}\left(C_{Z 1}, C_{Z 2}\right)=0.517, \operatorname{Corr}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)=0.417$. Model 3.

|  | $\Delta_{n}=5$ minutes |  |  |  |  |  | $\Delta_{n}=1$ minute |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\theta=1.5$ |  | $\theta=2.0$ |  | $\theta=2.5$ |  | $\theta=1.5$ |  | $\theta=2.0$ |  | $\theta=2.5$ |  |
|  | AN | LIN | AN | LIN | AN | LIN | AN | LIN | AN | LIN | AN | LIN |
|  | Panel A : $H_{0}^{1}:\left[C_{Z 1}, C_{Z 2}\right]_{T}=0$, Model 1 |  |  |  |  |  |  |  |  |  |  |  |
| $\alpha=10 \%$ | 9.7 | 10.6 | 10.6 | 12.6 | 9.7 | 10.3 | 10.2 | 9.7 | 10.0 | 10.2 | 9.8 | 10.2 |
| $\alpha=5 \%$ | 4.7 | 5.1 | 4.5 | 5.3 | 4.8 | 5.6 | 5.3 | 5.3 | 5.2 | 5.3 | 4.9 | 5.1 |
| $\alpha=1 \%$ | 0.9 | 1.1 | 0.9 | 1.2 | 0.9 | 1.1 | 1.1 | 1.1 | 1.2 | 1.1 | 1.0 | 1.0 |
| Panel B : $H_{0}^{2}:\left[C_{Z 1}, C_{X}\right]_{T}=0$, Model 1 |  |  |  |  |  |  |  |  |  |  |  |  |
| $\alpha=10 \%$ | 12.1 | 10.2 | 10.0 | 10.6 | 9.8 | 11.0 | 11.0 | 10.4 | 10.3 | 10.4 | 10.4 | 10.4 |
| $\alpha=5 \%$ | 6.2 | 5.0 | 4.5 | 5.2 | 4.6 | 5.4 | 5.5 | 5.4 | 5.2 | 5.1 | 5.2 | 5.3 |
| $\alpha=1 \%$ | 1.5 | 1.0 | 0.8 | 1.0 | 0.9 | 1.2 | 1.1 | 1.1 | 1.0 | 0.9 | 0.8 | 1.0 |
| Panel C : $H_{0}^{3}:\left[C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right]_{T}=0$, Model 2 |  |  |  |  |  |  |  |  |  |  |  |  |
| $\alpha=10 \%$ | 10.0 | 10.1 | 12.1 | 10.8 | 9.9 | 12.6 | 10.1 | 10.3 | 10.6 | 11.3 | 10.1 | 11.4 |
| $\alpha=5 \%$ | 5.0 | 6.3 | 5.1 | 6.3 | 5.1 | 6.7 | 5.5 | 5.5 | 5.3 | 5.9 | 5.2 | 6.0 |
| $\alpha=1 \%$ | 1.1 | 1.5 | 0.8 | 1.6 | 1.1 | 1.4 | 1.1 | 1.2 | 1.3 | 1.3 | 1.3 | 1.5 |

Table 8: Panel A contains the empirical rejection probabilities of the test of absence of dependence between IdioVols. Panel B contains the empirical rejection probabilities of the test of absence of dependence between the IdioVol and the market volatility. Panel C contains the empirical rejection probabilities of the test absence of dependence between residual IdioVols. $T=10$ years. $\alpha$ denotes the nominal size of the test.

## 7 Conclusion

We introduce an econometric framework for analysis of cross-sectional dependence in the IdioVols of assets using high frequency data. First, we provide bias-corrected estimators of standard measures of dependence between IdioVols, as well as the associated asymptotic theory. Second, we study an IdioVol Factor Model, in which we decompose the variation in IdioVols into two parts: the variation related to the systematic factors such as the market volatility, and the residual variation. We provide the asymptotic theory that allows us to test, for example, whether the residual (nonsystematic) components of the IdioVols exhibit cross-sectional dependence.

To provide the bias-corrected estimators and inference results, we develop a new asymptotic theory for general estimators of quadratic covariation of vector-valued (possibly) nonlinear transformations of the spot covariance matrices. This theoretical contribution is of its own interest, and can be applied in other contexts. For example, our results can be used to conduct inference for the cross-sectional dependence in asset betas.

We apply our methodology to the 30 Dow Jones Industrial Average components, and document strong cross-sectional dependence in their Idiosyncratic Volatilities. We consider two different sets of Idiosyncratic Volatility factors. We find that a single market volatility factor cannot fully account for the cross-sectional dependence in Idiosyncratic Volatilities, while this conclusion is reversed with additional industry volatility factors. For each model, we map out the network of dependencies in residual (non-systematic) Idiosyncratic Volatilities across all stocks.

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## Appendix

The Appendix collects all proofs (Sections A-D), and presents additional figures for the empirical application (in Section F).

## A Notation for Proofs

Our notation is similar to that of the proofs of Jacod and Rosenbaum (2015) whenever possible. Throughout, we denote by $K$ a generic constant, which may change from line to line. We let by convention $\sum_{i=a}^{a^{\prime}}=0$ when $a>a^{\prime}$. For simplicity, we omit the subscript $r$ for results involving only one object with this subscript. By the usual localization argument, there exists a $\pi$-integrable function $J$ on $E$ and a constant such that the stochastic processes in equations (26) and (27) satisfy

$$
\begin{equation*}
\|b\|,\|\widetilde{b}\|,\|c\|,\|\widetilde{c}\|, J \leq A,\|\delta(w, t, z)\|^{r} \leq J(z) \tag{A.1}
\end{equation*}
$$

For any càdlàg bounded process $Z$, we set

$$
\begin{aligned}
& \eta_{t, s}(Z)=\sqrt{\mathbb{E}\left(\sup _{0<u \leq s}\left\|Z_{t+u}-Z_{t}\right\|^{2} \mid \mathcal{F}_{i}^{n}\right)}, \text { and } \\
& \eta_{i, j}^{n}(Z)=\sqrt{\mathbb{E}\left(\sup _{0 \leq u \leq j \Delta_{n}}\left\|Z_{(i-1) \Delta_{n}+u}-Z_{(i-1) \Delta_{n}}\right\|^{2} \mid \mathcal{F}_{i}^{n}\right)} .
\end{aligned}
$$

For convenience, we decompose $Y_{t}$ as

$$
Y_{t}=Y_{0}+Y_{t}^{\prime}+\sum_{s \leq t} \Delta Y_{s}
$$

where $Y_{t}^{\prime}=\int_{0}^{t} b_{s}^{\prime} d s+\int_{0}^{t} \sigma_{s} d W_{s}$ and $b_{t}^{\prime}=b_{t}-\int \delta(t, z) 1_{\{\|\delta(t, z)\| \leq 1\}} \pi(d z)$.
Let $\widehat{C}_{i}^{\prime n}$ be the local estimator of the spot variance of the unobservable process $Y^{\prime}$, i.e.,

$$
\begin{equation*}
\widehat{C}_{i}^{\prime n}=\frac{1}{k_{n} \Delta_{n}} \sum_{u=0}^{k_{n}-1}\left(\Delta_{i+u}^{n} Y^{\prime}\right)\left(\Delta_{i+u}^{n} Y\right)^{\prime \top}=\left(\widehat{C}_{i}^{\prime n, g h}\right)_{1 \leq g, h \leq d} \tag{A.2}
\end{equation*}
$$

There is no jump truncation applied in the definition of $\widehat{C}_{i}^{\prime n}$ since the process $Y^{\prime}$ is continuous. Hence, it is more convenient to work with $\widehat{C}_{i}^{\prime n}$ rather than $\widehat{C}_{i}^{n}$ (defined in equation (14)).
We also define

$$
\begin{equation*}
\alpha_{i}^{n}=\left(\Delta_{i}^{n} Y^{\prime}\right)\left(\Delta_{i}^{n} Y^{\prime}\right)^{\top}-C_{i}^{n} \Delta_{n}, \quad \nu_{i}^{n}=\widehat{C}_{i}^{\prime n}-C_{i}^{n}, \quad \text { and } \lambda_{i}^{n}=\widehat{C}_{i+k_{n}}^{\prime n}-\widehat{C}_{i}^{\prime n} \tag{A.3}
\end{equation*}
$$

which satisfy

$$
\begin{equation*}
\nu_{i}^{n}=\frac{1}{k_{n} \Delta_{n}} \sum_{j=0}^{k_{n}-1}\left(\alpha_{i+j}^{n}+\left(C_{i+j}^{n}-C_{i}^{n}\right) \Delta_{n}\right) \text { and } \lambda_{i}^{n}=\nu_{i+k_{n}}-\nu_{i}^{n}+\Delta_{n}\left(C_{i+k_{n}}^{n}-C_{i}^{n}\right) \tag{A.4}
\end{equation*}
$$

The following multidimensional quantities will be used in the sequel

$$
\begin{aligned}
& \zeta(1)_{i}^{n}=\frac{1}{\Delta_{n}} \Delta_{i}^{n} Y^{\prime}\left(\Delta_{i}^{n} Y^{\prime}\right)^{\top}-C_{i-1}^{n}, \quad \zeta(2)_{i}^{n}=\Delta_{i}^{n} c \\
& \zeta^{\prime}(u)_{i}^{n}=\mathbb{E}\left(\zeta(u)_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right), \quad \zeta^{\prime \prime}(u)_{i}^{n}=\zeta(u)_{i}^{n}-\zeta^{\prime}(u)_{i}^{n}, \text { with } \quad \zeta^{r}(u)_{i}^{n}=\left(\zeta^{r}(u)_{i}^{n, g h}\right)_{1 \leq g, h \leq d}
\end{aligned}
$$

We also define, for $m \in\left\{0, \ldots, 2 k_{n}-1\right\}$ and $j, l \in \mathbb{Z}$,

$$
\varepsilon(1)_{m}^{n}=\left\{\begin{array}{ll}
-1 & \text { if } 0 \leq m<k_{n} \\
+1 & \text { if } k_{n} \leq m<2 k_{n},
\end{array}, \varepsilon(2)_{m}^{n}=\sum_{q=m+1}^{2 k_{n}-1} \varepsilon(1)_{q}^{n}=(m+1) \wedge\left(2 k_{n}-m-1\right),\right.
$$

For any $u, v, m, u^{\prime}, v^{\prime}$, we set

$$
\begin{gathered}
z_{u, v}^{n}= \begin{cases}1 / \Delta_{n} & \text { if } u=v=1 \\
1 & \text { otherwise, }\end{cases} \\
\lambda(u, v ; m)_{j, l}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{q=0 \vee(j-m)}^{(l-m-1) \vee\left(2 k_{n}-m-1\right)} \varepsilon(u)_{q}^{n} \varepsilon(u)_{q+m}^{n}, \quad \lambda(u, v)_{m}^{n}=\lambda(u, v ; m)_{0,2 k_{n}}^{n}, \\
M\left(u, v ; u^{\prime}, v^{\prime}\right)_{n}=z_{u, v}^{n} z_{u^{\prime}, v^{\prime}}^{n} \sum_{m=1}^{2 k_{n}-1} \lambda(u, v)_{m}^{n} \lambda\left(u^{\prime}, v^{\prime}\right)_{m}^{n} .
\end{gathered}
$$

Additionally, set

$$
\begin{align*}
\overline{A 11}(H, g h, u ; G, a b, v)_{T}^{n} & =\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\sum_{j=0}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right)\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \\
& =\lambda(u, v)_{0}^{n} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}, \tag{A.5}
\end{align*}
$$

and

$$
\begin{align*}
\overline{A 12}(H, g h, u ; G, a b, v)_{T}^{n} & =\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \sum_{m=1}^{(i-1) \wedge\left(2 k_{n}-1\right)} \sum_{j=0}^{\left(2 k_{n}-m-1\right)} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n} \\
& \times \zeta_{g h}(u)_{i-m}^{n} \zeta_{a b}(v)_{i}^{n} . \tag{A.6}
\end{align*}
$$

## B Auxiliary Lemmas and Theorems

This section presents useful auxiliary results, which are used in the proofs of Theorems 1 and 2. The results of this section are proved in Section E below.

First, we explain why we can assume, without loss of generality, that the derivatives of functions $H_{r}$ and $G_{r}$ are bounded, for $r=1, \ldots, \kappa$. Assumptions of Theorem 1 imply Lemma 2 of Li, Todorov, and Tauchen (2017a). Therefore, we can assume that the variables $\widehat{C}_{i \Delta_{n}}$ are bounded, uniformly over $i \in$ $\left\{0, \ldots,\left[T / \Delta_{n}\right]-k_{n}+1\right\}$, with probability approaching one. Using the spatial localization argument of Li, Todorov, and Tauchen (2016), which in turn uses the spatial localization argument of Li, Todorov, and Tauchen (2017a), we can assume that $H_{r}$ and $G_{r}$ are compactly supported without loss of generality. Hence, the derivatives of functions $H_{r}$ and $G_{r}$ are bounded, for $r=1, \ldots, \kappa$.

Theorem B1. Let $\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}\right.$ and $\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}} \quad\right.$ be the infeasible estimators obtained by replacing $\widehat{C}_{i}^{n}$ by $\widehat{C}_{i}^{\prime n}$ in the definition of $\left[H(\widehat{C), G}(C)]_{T}^{L I N}\right.$ and $\left[H(\widehat{C), G}(C)]_{T}^{A N}\right.$ in equations (18) and (16). As long as $3 / 4(2-r) \leq \varpi<\frac{1}{2}$, we have

$$
\Delta_{n}^{-1 / 4}\left(\left[H(\widehat{C), G}(C)]_{T}^{L I N}-\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}\right) \xrightarrow{\mathbb{P}} 0\right.\right.
$$

$$
\begin{equation*}
\text { and } \quad \Delta_{n}^{-1 / 4}\left(\left[H(\widehat{C), G}(C)]_{T}^{A N}-\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}\right) \xrightarrow{\mathbb{P}} 0\right.\right. \text {. } \tag{B.7}
\end{equation*}
$$

Theorem B1 allows, in particular, to focus on the derivation of the asymptotic distributions of $\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}} \quad\right.$ and $\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}\right.$. The next theorem connects the two estimators that we have introduced. To state the theorem, define

$$
\begin{aligned}
{\left[H(\widehat{C), G}(C)]_{T}^{A}\right.} & =\frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(( \partial _ { g h } H \partial _ { a b } G ) ( C _ { i } ^ { n } ) \left[\left(\widehat{C}_{i+k_{n}}^{\prime n, g h}-\widehat{C}_{i}^{\prime n, g h}\right)\left(\widehat{C}_{i+k_{n}}^{\prime n, a b}-\widehat{C}_{i}^{\prime n, a b}\right)\right.\right. \\
& \left.\left.-\frac{2}{k_{n}}\left(\widehat{C}_{i}^{\prime n, g a} \widehat{C}_{i}^{\prime n, h b}+\widehat{C}_{i}^{\prime n, g b} \widehat{C}_{i}^{\prime n, h a}\right)\right]\right) .
\end{aligned}
$$

with $C_{i}^{n}=C_{(i-1) \Delta_{n}}$, and the superscript $A$ stands for "approximated". For simplicity, we do not index the above quantity by a prime although it depends on $\widehat{C}_{i}^{\prime n}$ instead of $\widehat{C}_{i}^{n}$.

Theorem B2. Under the assumptions of Theorem 1, we have

$$
\begin{align*}
& \Delta_{n}^{-1 / 4}\left(\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}-\left[H(\widehat{C), G}(C)]_{T}^{A}\right) \xrightarrow{\mathbb{P}} 0\right.\right. \text { and } \\
& \Delta_{n}^{-1 / 4}\left(\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}-\left[H(\widehat{C), G}(C)]_{T}^{A}\right) \xrightarrow{\mathbb{P}} 0 .\right.\right. \tag{B.8}
\end{align*}
$$

Theorem B2 shows that the two estimators $\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}\right.$ and $\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}\right.$ can be approximated by a certain quantity with an error of approximation of order smaller than $\Delta_{n}^{-1 / 4}$.
Now, we decompose the approximated estimator as follows

$$
\begin{equation*}
\left[H(\widehat{C), G}(C)]_{T}^{(A)}=\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}-\left[H(\widehat{C), G}(C)]_{T}^{(A 2)},\right.\right.\right. \tag{B.9}
\end{equation*}
$$

with

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}=\frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}^{n}\right)\left(\widehat{C}_{i+k_{n}}^{\prime n, g h}-\widehat{C}_{i}^{\prime n, g h}\right)\left(\widehat{C}_{i+k_{n}}^{\prime n, a b}-\widehat{C}_{i}^{\prime n, a b}\right),\right.
$$

and

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 2)}=\frac{3}{k_{n}^{2}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i}^{\prime n}\right)\left(\widehat{C}_{i}^{\prime n, g a} \widehat{C}_{i}^{\prime n, h b}+\widehat{C}_{i}^{\prime \prime, g b} \widehat{C}_{i}^{\prime n, h a}\right) .\right.
$$

The following theorem holds:
Theorem B3. Under the assumptions of Theorem 1, we have

$$
\begin{gathered}
\frac{1}{\Delta_{n}^{1 / 4}}\left(\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}-\sum_{g, h, a, b=1}^{d} \sum_{u, v=1}^{2} \overline{A 11}(H, g h, u ; G, a b, v)_{T}^{n}+\overline{A 12}(H, g h, u ; G, a b, v)_{T}^{n}\right.\right. \\
\left.+\overline{A 12}(G, a b, v ; H, g h, u)_{T}^{n}\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0 .
\end{gathered}
$$

Lemma B1. For any càdlàg bounded process $Z$, for all $t, s>0, j, k \geq 0$, set $\eta_{t, s}=\eta_{t, s}(Z)$. Then,

$$
\Delta_{n} \mathbb{E}\left(\sum_{i=1}^{\left[t / \Delta_{n}\right]} \eta_{i, k_{n}}\right) \longrightarrow 0, \quad \Delta_{n} \mathbb{E}\left(\sum_{i=1}^{\left[t / \Delta_{n}\right]} \eta_{i, 2 k_{n}}\right) \longrightarrow 0
$$

$$
\mathbb{E}\left(\eta_{i+j, k} \mid \mathcal{F}_{i}^{n}\right) \leq \eta_{i, j+k} \quad \text { and } \quad \Delta_{n} \mathbb{E}\left(\sum_{i=1}^{\left[t / \Delta_{n}\right]} \eta_{i, 4 k_{n}}\right) \longrightarrow 0 .
$$

Lemma B2. Let $Z$ be a continuous Itô process with drift $b_{t}^{Z}$ and spot variance process $C_{t}^{Z}$, and set $\eta_{t, s}=$ $\eta_{t, s}\left(b^{Z}, c^{Z}\right)$. Then, the following bounds hold:

$$
\begin{align*}
& \left|\mathbb{E}\left(Z_{t} \mid \mathcal{F}_{0}\right)-t b_{0}^{Z}\right| \leq K t \eta_{0, t} \\
& \left|\mathbb{E}\left(Z_{t}^{j} Z_{t}^{k}-t C_{0}^{Z, j k} \mid \mathcal{F}_{0}\right)\right| \leq K t^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{0, t}\right) \\
& \left|\mathbb{E}\left(\left(Z_{t}^{j} Z_{t}^{k}-t C_{0}^{Z, j k}\right)\left(C_{t}^{Z, l m}-C_{0}^{Z, l m}\right) \mid \mathcal{F}_{0}\right)\right| \leq K t^{2} \\
& \left|\mathbb{E}\left(Z_{t}^{j} Z_{t}^{k} Z_{t}^{l} Z_{t}^{m} \mid \mathcal{F}_{0}\right)-\Delta_{n}^{2}\left(C_{0}^{Z, j k} C_{0}^{Z, l m}+C_{0}^{Z, j l} C_{0}^{Z, k m}+C_{0}^{Z, j m} C_{0}^{Z, k l}\right)\right| \leq K t^{5 / 2} \\
& \left|\mathbb{E}\left(Z_{t}^{j} Z_{t}^{k} Z_{t}^{l} \mid \mathcal{F}_{0}\right)\right| \leq K t^{2} \\
& \left|\mathbb{E}\left(\prod_{l=1}^{6} Z_{t}^{j_{l}} \mid \mathcal{F}_{0}\right)-\frac{\Delta_{n}^{3}}{6} \sum_{l<l^{\prime}} \sum_{k<k^{\prime}} \sum_{m<m^{\prime}} C_{0}^{Z, j_{l} j_{l^{\prime}}} C_{0}^{Z, j_{k} j_{k^{\prime}}} C_{0}^{Z, j_{m} j_{m^{\prime}}}\right| \leq K t^{7 / 2} \\
& \mathbb{E}\left(\sup _{w \in[0, s]}\left\|Z_{t+w}-Z_{t}\right\|^{q} \mid \mathcal{F}_{t}\right) \leq K_{q} s^{q / 2}, \text { and }\left\|\mathbb{E}\left(Z_{t+s}-Z_{t}\right) \mid \mathcal{F}_{t}\right\| \leq K s . \tag{B.10}
\end{align*}
$$

Lemma B3. Let $\zeta_{i}^{n}$ be a r-dimensional $\mathcal{F}_{i}^{n}$-measurable process satisfying $\left\|\mathbb{E}\left(\zeta_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right)\right\| \leq L^{\prime}$ and $\mathbb{E}\left(\left\|\zeta_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L_{q}$. Also, let $\varphi_{i}^{n}$ be a real-valued $\mathcal{F}_{i}^{n}$-measurable process with $\mathbb{E}\left(\left\|\varphi_{i+j-1}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L^{q}$ for $q \geq 2$ and $1 \leq j \leq 2 k_{n}-1$. Then,

$$
\mathbb{E}\left(\left\|\sum_{j=1}^{2 k_{n}-1} \varphi_{i+j-1}^{n} \zeta_{i+j}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K_{q} L^{q}\left(L_{q} k_{n}^{q / 2}+L^{\prime q} k_{n}^{q}\right)
$$

Lemma B4. Under the assumptions of Theorem 1, we have:

$$
\begin{aligned}
& \left\lvert\, \mathbb{E}\left(\lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \lambda_{i+2 k_{n}}^{n, g h} \lambda_{i+2 k_{n}}^{n, a b} \mid \mathcal{F}_{i}^{n}\right)-\frac{4}{k_{n}^{2}}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right)\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)\right. \\
& -\frac{4 \Delta_{n}}{3}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right) \bar{C}_{i}^{n, g h, a b}-\frac{4 \Delta_{n}}{3}\left(C_{i}^{n, g a} C_{i}^{n, h b}-C_{i}^{n, g b} C_{i}^{n, h a}\right) \bar{C}_{i}^{n, j k, l m} \\
& \left.-\frac{4\left(k_{n} \Delta_{n}\right)^{2}}{9} \bar{C}_{i}^{n, g h, a b} \bar{C}_{i}^{n, j k, l m} \right\rvert\, \leq K \Delta_{n}\left(\Delta_{n}^{1 / 8}+\eta_{i, 4 k_{n}}^{n}\right) .
\end{aligned}
$$

Lemma B5. Under the assumptions of Theorem 1, we have:

$$
\begin{array}{r}
\left|\mathbb{E}\left(\nu_{i}^{n, j k} \nu_{i}^{n, l m} \nu_{i}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}^{n}\right), \\
\left|\mathbb{E}\left(\nu_{i}^{n, j k} \nu_{i}^{n, l m}\left(c_{i+k_{n}}^{n, g h}-c_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}^{n}\right), \\
\left|\mathbb{E}\left(\nu_{i}^{n, j k}\left(c_{i+k_{n}}^{n, l m}-c_{i}^{n, l m}\right)\left(c_{i+k_{n}}^{n, g h}-c_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}^{n}\right), \\
\left|\mathbb{E}\left(\nu_{i}^{n, j k} \lambda_{i}^{n, l m} \lambda_{i}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, 2 k_{n}}^{n}\right), \tag{B.15}
\end{array}
$$

$$
\begin{equation*}
\left|\mathbb{E}\left(\lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \lambda_{i}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, 2 k_{n}}^{n}\right) \tag{B.16}
\end{equation*}
$$

Lemma B6. Under the assumptions of Theorem 1, we have:

$$
\begin{align*}
& \frac{1}{\Delta_{n}^{1 / 4}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \rho_{g h}(u, v)_{i}^{n} \zeta_{a b}^{\prime}(v)_{i}^{n} \stackrel{\mathbb{P}}{\Longrightarrow} 0, \quad \forall(u, v)  \tag{B.17}\\
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\overline{A 11}(H, g h, u ; G, a b, v)-\int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right) \bar{C}_{t}^{g h, a b} d t\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \text { when }(u, v)=(2,2)  \tag{B.18}\\
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\overline{A 11}(H, g h, u ; G, a b, v)-\frac{3}{\theta^{2}} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right)\left(C_{t}^{g a} C_{t}^{h b}+C_{t}^{g b} C_{t}^{h a}\right) d t\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0  \tag{B.19}\\
& w h e n \\
& \frac{1}{\Delta_{n}^{1 / 4}} \overline{A 11}(H, v)=(1,1), \tag{B.20}
\end{align*}
$$

## C Proof of Theorem 1

We now prove Theorem 1. By Theorem B3, we have

$$
\begin{aligned}
\frac{1}{\Delta_{n}^{1 / 4}}\left(\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}\right.\right. & -\sum_{g, h, a, b=1}^{d} \sum_{u, v=1}^{2} \overline{A 11}(H, g h, u ; G, a b, v)_{T}^{n}+\overline{A 12}(H, g h, u ; G, a b, v)_{T}^{n} \\
& \left.+\overline{A 12}(G, a b, v ; H, g h, u)_{T}^{n}\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0
\end{aligned}
$$

Recalling the definition of $\overline{A 12}(H, g h, u ; G, a b, v)_{T}^{n}$ from equation (A.6), Lemma B6 implies that

$$
\begin{align*}
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\left[H(\widehat{C), G}(C)]_{T}^{(A)}-[H(C), G(C)]_{T}-\frac{3}{2 k_{n}^{3}} \sum_{g, h, a, b}^{d} \sum_{u, v=1}^{2} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\right.\right. \\
& \left.\left[\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \rho_{g h}(u, v)_{i}^{n} \zeta_{a b}^{\prime \prime}(v)_{i}^{n}+\left(\partial_{a b} H \partial_{g h} G\right)\left(C_{i-2 k_{n}}^{n}\right) \rho_{a b}(v, u)_{i}^{n} \zeta_{g h}^{\prime \prime}(v)_{i}^{n}\right]\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \tag{C.21}
\end{align*}
$$

Next, define

$$
\begin{aligned}
\xi(H, g h, u ; G, a b, v)_{i}^{n} & =\frac{1}{\Delta_{n}^{1 / 4}}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \rho_{g h}(u, v)_{i}^{n} \zeta_{a b}^{\prime \prime}(v)_{i}^{n} \\
Z(H, g h, u ; G, a b, v)_{t}^{n} & =\Delta_{n}^{1 / 4} \sum_{i=2 k_{n}}^{\left[t / \Delta_{n}\right]} \xi(H, g h, u ; G, a b, v)_{i}^{n}
\end{aligned}
$$

Notice that (C.21) implies

$$
\begin{align*}
& \frac{1}{\Delta_{n}^{1 / 4}}\left([ H ( \widehat { C ) , G } ( C ) ] _ { T } ^ { ( A ) } - [ H ( C ) , G ( C ) ] _ { T } ) \stackrel { \mathcal { L } } { = } \sum _ { g , h , a , b = 1 } ^ { d } \sum _ { u , v = 1 } ^ { 2 } \frac { 1 } { \Delta _ { n } ^ { 1 / 4 } } \left(Z(H, g h, u ; G, a b, v)_{T}^{n}\right.\right. \\
& \left.+Z(H, a b, v ; G, g h, u)_{T}^{n}\right) \tag{C.22}
\end{align*}
$$

Next, observe that to derive the asymptotic distribution of $\left(\left[H_{1}\left(\widehat{C), G_{1}}(C)\right]_{T}^{(A)}, \ldots,\left[H_{\kappa}\left(\widehat{C), G_{\kappa}}(C)\right]_{T}^{(A)}\right)\right.\right.$, it suffices to study the joint asymptotic behavior of the family of processes $\frac{1}{\Delta_{n}^{1 / 4}} Z(H, g h, u ; G, a b, v)_{T}^{n}$. Notice that $\xi(H, g h, u ; G, a b, v)_{i}^{n}$ are martingale increments relative to the discrete filtration $\left(\mathcal{F}_{i}^{n}\right)$. Therefore, to obtain the joint asymptotic distribution of $\frac{1}{\Delta_{n}^{1 / 4}} Z(H, g h, u ; G, a b, v)_{T}^{n}$, it is enough to prove the following three properties:

$$
\begin{align*}
& A\left((H, g h, u ; G, a b, v),\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{t}^{n}=\sum_{i=2 k_{n}}^{\left[t / \Delta_{n}\right]} \mathbb{E}\left(\xi(H, g h, u ; G, a b, v)_{i}^{n} \xi\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right) \\
& \quad \stackrel{\mathbb{P}}{\Longrightarrow} A\left((H, g h, u ; G, a b, v),\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{t}  \tag{C.23}\\
& \sum_{i=2 k_{n}}^{\left[t / \Delta_{n}\right]} \mathbb{E}\left(\left|\xi(H, g h, u ; G, a b, v)_{i}^{n}\right|^{4} \mid \mathcal{F}_{i-1}^{n}\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0, \text { and }  \tag{C.24}\\
& B(N ; H, g h, u ; G, a b, v)_{t}^{n}:=\sum_{i=2 k_{n}}^{\left[t / \Delta_{n}\right]} \mathbb{E}\left(\xi(H, g h, u ; G, a b, v)_{i}^{n} \Delta_{i}^{n} N \mid \mathcal{F}_{i-1}^{n}\right) \stackrel{\mathbb{P}}{\Longrightarrow} 0 \tag{C.25}
\end{align*}
$$

for all $t>0$, all $(H, g h, u ; G, a b, v),\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)$ and all martingales $N$ which are either bounded and orthogonal to $W$, or equal to one component $W^{j}$.
Since the derivatives of $H_{r}$ and $G_{r}$ are bounded, equations (C.24) and (C.25) can be proved by an extension of (B.105) and (B.106) in Aït-Sahalia and Jacod (2014) to multivariate processes.
Next, define

$$
V_{a b}^{a^{\prime} b^{\prime}}\left(v, v^{\prime}\right)_{t}= \begin{cases}\left(C_{t}^{a a^{\prime}} C_{t}^{b b^{\prime}}+C_{t}^{a b^{\prime}} C_{t}^{b a^{\prime}}\right) & \text { if } \quad\left(v, v^{\prime}\right)=(1,1) \\ \bar{C}_{t}^{a b, a^{\prime} b^{\prime}} & \text { if } \quad\left(v, v^{\prime}\right)=(2,2) \\ 0 & \text { otherwise },\end{cases}
$$

and

$$
\bar{V}_{g h}^{g^{\prime} h^{\prime}}\left(u, u^{\prime}\right)_{t}= \begin{cases}\left(C_{t}^{g g^{\prime}} C_{t}^{h h^{\prime}}+C_{t}^{g h^{\prime}} C_{t}^{h g^{\prime}}\right) & \text { if } \quad\left(u, u^{\prime}\right)=(1,1) \\ \bar{C}_{t}^{g h, g^{\prime} h^{\prime}} & \text { if } \quad\left(u, u^{\prime}\right)=(2,2) \\ 0 & \text { otherwise }\end{cases}
$$

Using again the boundedness of the derivatives of $H_{r}$ and $G_{r}$, we can show that

$$
\begin{aligned}
& A\left((H, g h, u ; G, a b, v),\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{t}= \\
& M\left(u, v ; u^{\prime}, v^{\prime}\right) \int_{0}^{t}\left(\partial_{g h} H \partial_{a b} G \partial_{g^{\prime} h^{\prime}} H \partial_{a^{\prime} b^{\prime}} G\right)\left(C_{s}\right) V_{a b}^{a^{\prime} b^{\prime}}\left(v, v^{\prime}\right)_{s} \bar{V}_{g h}^{g^{\prime} h^{\prime}}\left(u, u^{\prime}\right)_{s} d s
\end{aligned}
$$

with

$$
M\left(u, v ; u^{\prime}, v^{\prime}\right)= \begin{cases}3 / \theta^{3} & \text { if } \quad\left(u, v ; u^{\prime}, v^{\prime}\right)=(1,1 ; 1,1) \\ 3 / 4 \theta & \text { if } \quad\left(u, v ; u^{\prime}, v^{\prime}\right)=(1,2 ; 1,2),(2,1 ; 2,1) \\ 151 \theta / 280 & \text { if } \quad\left(u, v ; u^{\prime}, v^{\prime}\right)=(2,2 ; 2,2) \\ 0 & \text { otherwise. }\end{cases}
$$

Therefore, we have

$$
\begin{aligned}
& A\left((H, g h, u ; G, a b, v),\left(H^{\prime}, g^{\prime} h^{\prime}, u^{\prime} ; G^{\prime}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{T}= \\
& \left\{\begin{array}{lr}
\frac{3}{\nu^{3}} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G \partial_{g^{\prime} h^{\prime}} H^{\prime} \partial_{a^{\prime} b^{\prime}} G^{\prime}\right)\left(C_{t}\right)\left(C_{t}^{g g^{\prime}} C_{t}^{h h^{\prime}}+C_{t}^{g h^{\prime}} C_{t}^{h g^{\prime}}\right)\left(C_{t}^{a a^{\prime}} C_{t}^{b b^{\prime}}+C_{t}^{a b^{\prime}} C_{t}^{b a^{\prime}}\right) d t, \\
\frac{3}{4 \nu} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G \partial_{g^{\prime} h^{\prime}} H^{\prime} \partial_{a^{\prime} b^{\prime}} G^{\prime}\right)\left(C_{t}\right)\left(C_{t}^{g g^{\prime}} C_{t}^{h h^{\prime}}+C_{t}^{g h^{\prime}} C_{t}^{h g^{\prime}}\right) \bar{C}_{t}^{a b, a^{\prime} b^{\prime} b^{\prime}} d t, \text { if }\left(u, v ; u^{\prime}, v^{\prime}\right)=(1,1 ; 1,1) \\
\left.\frac{3}{4 \nu} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G \partial_{g^{\prime} h^{\prime}} H^{\prime} \partial_{a^{\prime} b^{\prime}} G^{\prime}\right)\left(C_{t}\right)\left(C_{t}^{a a^{\prime}} C_{t}^{b b^{\prime}}+C_{t}^{a b^{\prime}} C_{s}^{b a^{\prime}}\right) \bar{t}_{s}^{g h, g^{\prime} h^{\prime}} d t, 2 ; 1,2\right) \\
\frac{151 \nu}{280} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G \partial_{g^{\prime} h^{\prime}} H^{\prime} \partial_{a^{\prime} b^{\prime}} G^{\prime}\right)\left(C_{t}\right) \bar{C}_{s}^{a b, a^{\prime} b^{\prime}} \bar{C}_{t}^{g h, g^{\prime} h^{\prime}} d t, & \text { if }\left(u, v ; u^{\prime}, v^{\prime}\right)=(2,1 ; 2,1) \\
0 & \text { otherwise. }
\end{array}\right.
\end{aligned}
$$

Using equation (C.22), we deduce that the asymptotic covariance between $\left[H_{r}\left(\widehat{C), G}_{r}(C)\right]_{T}^{(A)}\right.$ and $\left[H_{s}\left(\widehat{C), G_{s}}(C)\right]_{T}^{(A)}\right.$ is given by

$$
\begin{aligned}
& \sum_{g, h, a, b=1}^{d} \sum_{g^{\prime}, h^{\prime}, a^{\prime}, b^{\prime}=1}^{d} \sum_{u, v, u^{\prime}, v^{\prime}=1}^{2}\left(A\left(\left(H_{r}, g h, u ; G_{r}, a b, v\right),\left(H_{s}, g^{\prime} h^{\prime}, u^{\prime} ; G_{s}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{T}\right. \\
& +A\left(\left(H_{r}, g h, u ; G_{r}, a b, v\right),\left(H_{s}, a^{\prime} b^{\prime}, v^{\prime} ; G_{s}, g^{\prime} h^{\prime}, u^{\prime}\right)\right)_{T} \\
& +A\left(\left(H_{r}, a b, v ; G_{r}, g h, u\right),\left(H_{s}, g^{\prime} h^{\prime}, u^{\prime} ; G_{s}, a^{\prime} b^{\prime}, v^{\prime}\right)\right)_{T} \\
& \left.+A\left(\left(H_{r}, a b, v ; H_{r}, g h, u\right),\left(H_{s}, a^{\prime} b^{\prime}, v^{\prime} ; G_{s}, g^{\prime} h^{\prime}, u^{\prime}\right)\right)_{T}\right)
\end{aligned}
$$

The above expression can be rewritten as

$$
\begin{aligned}
& \quad \sum_{g, h, a, b=1}^{d} \sum_{j, k, l, m=1}^{d}\left(\frac { 6 } { \theta ^ { 3 } } \int _ { 0 } ^ { T } ( \partial _ { g h } H _ { r } \partial _ { a b } G _ { r } \partial _ { j k } H _ { s } \partial _ { l m } G _ { s } ( C _ { t } ) ) \left[\left(C_{t}^{g j} C_{t}^{h k}+C_{t}^{g k} C_{t}^{h j}\right)\left(C_{t}^{a l} C_{t}^{b m}+C_{t}^{a m} C_{t}^{b l}\right)\right.\right. \\
& \left.+\left(C_{t}^{a j} C_{t}^{b k}+C_{t}^{a k} C_{t}^{b j}\right)\left(C_{t}^{g l} C_{t}^{h m}+C_{t}^{g m} C_{t}^{h l}\right)\right] d t \\
& +\frac{151 \theta}{140} \int_{0}^{t}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(C_{t}\right)\right)\left[\bar{C}^{g h, j k} \bar{C}^{a b, l m}+\bar{C}^{a b, j k} \bar{C}^{g h, l m}\right] d t \\
& +\frac{3}{2 \theta} \int_{0}^{t}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\left(C_{t}\right)\right)\left[\left(C_{t}^{g j} C_{t}^{h k}+C_{t}^{g k} C_{t}^{h j}\right) \bar{C}_{t}^{a b, l m}+\left(C_{t}^{a l} C_{t}^{b m}+C_{t}^{a m} C_{t}^{b l}\right) \bar{C}_{t}^{g h, j k}\right. \\
& + \\
& \left.\left.+\left(C_{t}^{g l} C_{s}^{h m}+C_{t}^{g m} C_{s}^{h l}\right) \bar{C}_{t}^{a b, j k}+\left(C_{t}^{a j} C_{t}^{b k}+C_{t}^{a k} C_{t}^{b j}\right) \bar{C}_{t}^{g h, l m}\right] d t\right)
\end{aligned}
$$

which completes the proof.

## D Proof of Theorem 2

Using boundedness of the derivatives of $H_{r}, G_{r}, H_{s}$ and $G_{s}$ and Theorem 2.2 in Jacod and Rosenbaum (2015), one can show that

$$
\frac{6}{\theta^{3}} \widehat{\Omega}_{T}^{r, s,(1)} \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(1)}
$$

Next, by equation (3.27) in Jacod and Rosenbaum (2015), we have

$$
\frac{3}{2 \theta}\left[\widehat{\Omega}_{T}^{r, s,(3)}-\frac{6}{\theta} \widehat{\Omega}_{T}^{r, s,(1)}\right] \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(3)}
$$

Finally, to show that

$$
\frac{151 \theta}{140} \frac{9}{4 \theta^{2}}\left[\widehat{\Omega}_{T}^{r, s,(2)}+\frac{4}{\theta^{2}} \widehat{\Omega}_{T}^{r, s,(1)}-\frac{4}{3} \widehat{\Omega}_{T}^{r, s,(3)}\right] \xrightarrow{\mathbb{P}} \Sigma_{T}^{r, s,(2)}
$$

we first observe that the approximation error induced by replacing $\widehat{C}_{i}^{n}$ by $\widehat{C}_{i}^{\prime n}$ in Theorem 2 is negligible. For $1 \leq g, h, a, b, j, k, l, m \leq d$ and $1 \leq r, s \leq d$, we define

$$
\begin{aligned}
\widehat{W}_{T}^{n} & =\sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{g h} H_{s} \partial_{l m} G_{s}\right)\left(\widehat{C}_{i}^{n}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i+2 k_{n}}^{n, a b} \lambda_{i+2 k_{n}}^{n, l m}, \\
\widehat{w}(1)_{i}^{n} & =\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{i}^{n}\right) \mathbb{E}\left(\lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i+2 k_{n}}^{n, a b} \lambda_{i+2 k_{n}}^{n, l m} \mid \mathcal{F}_{i}^{n}\right), \\
\widehat{w}(2)_{i}^{n} & =\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{i}^{n}\right)\left(\lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i+2 k_{n}}^{n, a b} \lambda_{i+2 k_{n}}^{n, l m}-\mathbb{E}\left(\lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i+2 k_{n}}^{n, a b} \lambda_{i+2 k_{n}}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)\right), \\
\widehat{w}(3)_{i}^{n} & =\left(\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(\widehat{C}_{i}^{n}\right)-\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{i}^{n}\right)\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i+2 k_{n}}^{n, a b} \lambda_{i+2 k_{n}}^{n, l m}, \\
\widehat{W}(u)_{t}^{n} & =\sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1} \widehat{w}_{i}(u), u=1,2,3 .
\end{aligned}
$$

Now, note that we also have $\widehat{W}_{t}^{n}=\widehat{W}(1)_{t}^{n}+\widehat{W}(2)_{t}^{n}+\widehat{W}(3)_{t}^{n}$. By Taylor expansion and using repeatedly the boundedness of $C_{t}$, we obtain

$$
\left|\widehat{w}(3)_{i}^{n}\right| \leq K\left\|\nu_{i}^{n}\right\|\left\|\lambda_{i}^{n}\right\|^{2}\left\|\lambda_{i+2 k_{n}}^{n}\right\|^{2}
$$

which implies $\mathbb{E}\left(\left|\widehat{w}(3)_{i}^{n}\right|\right) \leq K \Delta_{n}^{5 / 4}$ and hence $\widehat{W}(3)_{t}^{n} \xrightarrow{\mathbb{P}} 0$. Using Cauchy-Schwartz inequality and the bound $\mathbb{E}\left(\left\|\lambda_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}^{q / 4}$, we have $\mathbb{E}\left(\left|\widehat{w}(2)_{i}^{n}\right|^{2}\right) \leq K \Delta_{n}^{2}$. Observing furthermore that $\widehat{w}(2)_{i}^{n}$ is $\mathcal{F}_{i+4 k_{n}}$-measurable, Lemma B. 8 in Aït-Sahalia and Jacod (2014) implies $\widehat{W}(2)_{t}^{n} \xrightarrow{\mathbb{P}} 0$. Next, define

$$
\begin{aligned}
& w_{i}^{n}=\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{i}^{n}\right)\left[\frac{4}{k_{n}^{2} \Delta_{n}}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right)\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)\right. \\
& \quad+\frac{4}{3}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right) \bar{C}_{i}^{n, g h, a b}+\frac{4}{3}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right) \bar{C}_{i}^{n, j k, l m} \\
& \left.\quad+\frac{4\left(k_{n}^{2} \Delta_{n}\right)}{9} \bar{C}_{i}^{n, g h, a b} \bar{C}_{i}^{n, j k, l m}\right] \\
& W_{T}^{n}=\Delta_{n} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1} w_{i}^{n} .
\end{aligned}
$$

Using the cadlag property of $c$ and $\bar{C}, k_{n} \sqrt{\Delta_{n}} \rightarrow \theta$, and the Riemann integral convergence, we conclude that $W_{T}^{n} \xrightarrow{\mathbb{P}} W_{T}$ where

$$
\begin{aligned}
& W_{T}=\int_{0}^{T}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{t}\right)\left[\frac{4}{\theta^{2}}\left(C_{t}^{g a} C_{t}^{h b}+C_{t}^{g b} C_{t}^{h a}\right)\left(C_{t}^{j l} C_{t}^{k m}+C_{t}^{j m} C_{t}^{k l}\right)\right. \\
& \left.+\frac{4}{3}\left(C_{t}^{j l} C_{t}^{k m}+C_{t}^{j m} C_{t}^{k l}\right) \bar{C}_{t}^{g h, a b}+\frac{4}{3}\left(C_{t}^{g a} C_{i}^{h b}+C_{t}^{g b} C_{t}^{h a}\right) \bar{C}_{t}^{j k, l m}+\frac{4 \theta^{2}}{9} \bar{C}_{t}^{g h, a b} \bar{C}_{t}^{j k, l m}\right] d t
\end{aligned}
$$

In addition, by Lemma B4, it holds that

$$
\mathbb{E}\left(\left|\widehat{W}(1)_{T}^{n}-W_{T}^{n}\right|\right) \leq \Delta_{n} \mathbb{E}\left(\sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\Delta_{n}^{1 / 8}+\eta_{i, 4 k_{n}}\right)\right)
$$

Hence, by the third result of Lemma B1 we have $\widehat{W}_{T}^{n} \xrightarrow{\mathbb{P}} W_{t}$, from which it follows that

$$
\begin{aligned}
& \frac{9}{4 \theta^{2}}\left[\widehat{W}(1)_{T}^{n}+\frac{4}{k_{n}^{2}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(\widehat{C}_{i}^{n}\right)\left[C_{i}^{n}(j k, l m) C_{i}^{n}(g h, a b)\right]\right. \\
& -\frac{2}{k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(\widehat{C}_{i}^{n}\right) C_{i}^{n}(g h, a b) \lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \\
& \left.-\frac{2}{k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-4 k_{n}+1}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(\widehat{C}_{i}^{n}\right) C_{i}^{n}(j k, l m) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b}\right] \\
& \xrightarrow{\mathbb{P}} \int_{0}^{T}\left(\partial_{g h} H_{r} \partial_{a b} G_{r} \partial_{j k} H_{s} \partial_{l m} G_{s}\right)\left(C_{t}\right) \bar{C}_{t}^{g h, a b} \bar{C}_{t}^{j k, l m} d t .
\end{aligned}
$$

The result follows from the above convergence, the already invoked symmetry argument, and straightforward calculations.

## E Proofs of Auxiliary Lemmas and Theorems

This section is devoted to the proofs of the auxiliary theorems and lemmas (listed in Section B) that were used to prove Theorem 1 and Theorem 2.

## E. 1 Proof of Theorem B1

To show this result, let us define the functions

$$
\begin{aligned}
R(x, y) & =\sum_{g, h, a, b=1}^{d}\left(\partial_{g h} H \partial_{a b} G\right)(x)\left(y^{g h}-x^{g h}\right)\left(y^{a b}-x^{a b}\right) \\
S(x, y) & =(H(y)-H(x))(G(y)-G(x)) \\
U(x) & =\sum_{g, h, a, b=1}^{d}\left(\partial_{g h} H \partial_{a b} G\right)(x)\left(x^{g a} x^{h b}+x^{g b} x^{h a}\right)
\end{aligned}
$$

for any $\mathbb{R}^{d} \times \mathbb{R}^{d}$ matrices $x$ and $y$. The following decompositions hold,

$$
\begin{aligned}
& {\left[H(\widehat{C), G}(C)]_{T}^{A N}-\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}\right.\right.} \\
& \quad=\frac{3}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left[\left(S\left(\widehat{C}_{i}^{n}, \widehat{C}_{i+k_{n}}^{n}\right)-S\left(\widehat{C}_{i}^{\prime n}, \widehat{C}_{i+k_{n}}^{\prime n}\right)\right)-\frac{2}{k_{n}}\left(U\left(\widehat{C}_{i}^{n}\right)-U\left(\widehat{C}_{i}^{\prime} n\right)\right)\right] \\
& {\left[H(\widehat{C), G}(C)]_{T}^{L I N}-\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}\right.\right.} \\
& \quad=\frac{3}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left[\left(R\left(\widehat{C}_{i}^{n}, \widehat{C}_{i+k_{n}}^{n}\right)-R\left(\widehat{C}_{i}^{\prime n}, \widehat{C}_{i+k_{n}}^{\prime n}\right)\right)-\frac{2}{k_{n}}\left(U\left(\widehat{C}_{i}^{n}\right)-U\left(\widehat{C}_{i}^{\prime n}\right)\right)\right] .
\end{aligned}
$$

By (3.11) in Jacod and Rosenbaum (2015), there exists a sequence of real numbers $a_{n}$ converging to zero such that

$$
\begin{equation*}
\mathbb{E}\left(\left\|\widehat{C}_{i}^{n}-\widehat{C}_{i}^{\prime n}\right\|^{q}\right) \leq K_{q} a_{n} \Delta_{n}^{(2 q-r) \varpi+1-q}, \text { for any } q>0 \tag{E.26}
\end{equation*}
$$

Since $H$ and $G$ are three times continuously differentiable with bounded derivatives, the functions $R$ and $S$ are continuously differentiable and satisfy

$$
\begin{align*}
\|\partial J(x, y)\| & \leq K \text { for } 1 \leq g, h, a, b \leq d \text { and } J \in\{S, R\}  \tag{E.27}\\
\|\partial U(x)\| & \leq K \tag{E.28}
\end{align*}
$$

where $\partial J$ (respectively, $\partial U$ ) is a vector that collects the first order partial derivatives of the function $J$ (respectively, $U$ ) with respect to all the elements of $(x, y)$ (respectively, $x$ ). Using the Taylor expansion, (E.27) and (E.28), it holds that, for $J \in\{S, R\}$,

$$
\begin{aligned}
\left|J\left(\widehat{C}_{i}^{n}, \widehat{C}_{i+k_{n}}^{n}\right)-J\left(\widehat{C}_{i}^{\prime n}, \widehat{C}_{i+k_{n}}^{\prime n}\right)\right| & \leq K\left(\left\|\widehat{C}_{i}^{n}-\widehat{C}_{i}^{\prime n}\right\|+\left\|\widehat{C}_{i+k_{n}}^{n}-\widehat{C}_{i+k_{n}}^{\prime n}\right\|\right) \text { and } \\
\left|U\left(\widehat{C}_{i}^{n}\right)-U\left(\widehat{C}_{i}^{\prime n}\right)\right| & \leq K\left(\left\|\widehat{C}_{i}^{n}-\widehat{C}_{i}^{\prime n}\right\|\right) .
\end{aligned}
$$

By equation (E.26), the following condition is sufficient for Theorem B1 to hold:

$$
(2-r) \varpi-\frac{3}{4} \geq 0
$$

Using the fact that $0<\varpi<\frac{1}{2}$, we can see that Theorem B1 holds when $3 / 4(2-r) \leq \varpi<\frac{1}{2}$, which completes the proof.

## E. 2 Proof of Theorem B2

Note that we have

$$
\begin{aligned}
& {\left[H(\widehat{C), G}(C)]_{T}^{L I N^{\prime}}-\left[H(\widehat{C), G}(C)]_{T}^{A}=\frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \psi_{i}^{n}(g, h, a, b),\right.\right.} \\
& {\left[H(\widehat{C), G}(C)]_{T}^{A N^{\prime}}-\left[H(\widehat{C), G}(C)]_{T}^{A}=\frac{3}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\chi_{i}^{n}-\sum_{g, h, a, b=1}^{d}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i}^{n}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b}\right),\right.\right.}
\end{aligned}
$$

with

$$
\begin{aligned}
& \psi_{i}^{n}(g, h, a, b)=\left(\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i}^{\prime n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i}^{n}\right)\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b} \\
& \chi_{i}^{n}=\left(H\left(\widehat{C}_{i+k_{n}}^{\prime n}\right)-H\left(\widehat{C}_{i}^{\prime n}\right)\right)\left(G\left(\widehat{C}_{i+k_{n}}^{\prime n}\right)-G\left(\widehat{C}_{i}^{\prime n}\right)\right)
\end{aligned}
$$

By Taylor expansion, we have

$$
\begin{aligned}
& \left(\partial_{g h} S \partial_{a b} G\right)\left(\widehat{C}_{i}^{\prime n}\right)-\left(\partial_{g h} S \partial_{a b} G\right)\left(C_{i}^{n}\right)=\sum_{x, y=1}^{d}\left(\partial_{x y, g h}^{2} S \partial_{a b} G+\partial_{x y, a b}^{2} G \partial_{g h} S\right)\left(C_{i}^{n}\right) \nu_{i}^{n, x y} \\
& +\frac{1}{2} \sum_{j, k, x, y=1}^{d}\left(\partial_{j k, x y, g h}^{3} S \partial_{a b} G+\partial_{x y, g h}^{2} S \partial_{j k, a b}^{2} G+\partial_{j k, x y, a b}^{3} G \partial_{g h} S+\partial_{x y, a b}^{2} G \partial_{j k, g h}^{2} S\right)\left(\widetilde{c}_{i}^{n}\right) \nu_{i}^{n, x y} \nu_{i}^{n, j k}
\end{aligned}
$$

and

$$
S\left(\widehat{C}_{i+k_{n}}^{\prime n}\right)-S\left(\widehat{C}_{i}^{\prime n}\right)=\sum_{g h} \partial_{g h} S\left(C_{i}^{n}\right) \lambda_{i}^{n, g h}+\sum_{j, k, g, h} \partial_{j k, g h}^{2} S\left(C_{i}^{n}\right) \lambda_{i}^{n, g h} \nu_{i}^{n, j k}
$$

$$
\begin{aligned}
& +\frac{1}{2} \sum_{x, y, g, h} \partial_{x y, g h}^{2} S\left(C_{i}^{n}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, x y}+\frac{1}{2} \sum_{x, y, j, k, g, h} \partial_{x y, j k, g h}^{3} S\left(C C_{i}^{n, S}\right) \lambda_{i}^{n, g h} \nu_{i}^{n, x y} \nu_{i}^{n, j k} \\
& +\frac{1}{6} \sum_{j, k, x, y, g, h} \partial_{j k, x y, g h}^{3} S\left(C_{i}^{n, S}\right) \lambda_{i}^{n, j k} \lambda_{i}^{n, g h} \lambda_{i}^{n, x y}
\end{aligned}
$$

for $S \in\{H, G\}, \widetilde{c}_{i}^{n}=\pi C_{i}^{n}+(1-\pi) \widehat{C}_{i}^{\prime n}, C_{i}^{n, S}=\pi_{S} \widehat{C}_{i}^{\prime n}+\left(1-\pi_{S}\right) \widehat{C}_{i+k_{n}}^{\prime n}, C C_{i}^{n, S}=\mu_{S} C_{i}^{n}+\left(1-\mu_{S}\right) \widehat{C}_{i}^{\prime n}$ for $\pi, \pi_{H}, \mu_{H}, \pi_{G}, \mu_{G} \in[0,1]$. Although $\widetilde{c}_{i}^{n}$ and $\pi$ depend on $g, h, a$, and $b$, we do not emphasize this in our notation to simplify the exposition.
Set $\mathcal{F}_{i}^{n}=\mathcal{F}_{(i-1) \Delta_{n}}$. By (4.10) in Jacod and Rosenbaum (2013) we have

$$
\begin{equation*}
\mathbb{E}\left(\left\|\alpha_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{q} \text { for all } q \geq 0 \text { and } \mathbb{E}\left(\left|\sum_{j=0}^{k_{n}-1} \alpha_{i+j}^{n}\right|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{q} k_{n}^{q / 2} \text { whenever } q \geq 2 \tag{E.29}
\end{equation*}
$$

Combining (E.29), (A.4), (B.10) with $Z=c$ and the Hölder inequality yields for $q \geq 2$,

$$
\begin{equation*}
\mathbb{E}\left(\left\|\nu_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta^{q / 4}, \text { and } \mathbb{E}\left(\left\|\lambda_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta^{q / 4} \tag{E.30}
\end{equation*}
$$

The bound in the first equation of (E.30) is tighter than that in (4.11) of Jacod and Rosenbaum (2015) due to the absence of volatility jumps. This tighter bound will be useful later in deriving the asymptotic distribution for the approximated estimator. By the boundedness of $C_{t}$ and the derivatives of $H$ and $G$,

$$
\begin{equation*}
\left|\left(\partial_{j k, x y, a b}^{3} G \partial_{g h} H+\partial_{x y, g h}^{2} H \partial_{j k, a b}^{2} G\right)\left(\widetilde{c}_{i}^{n}\right) \nu_{i}^{n, x y} \nu_{i}^{n, j k} \lambda_{i}^{n, g h} \lambda_{i}^{n, a b}\right| \leq K\left\|\nu_{i}^{n}\right\|^{2}\left\|\lambda_{i}^{n}\right\|^{2} \tag{E.31}
\end{equation*}
$$

Using the Taylor expansion, we have

$$
\begin{aligned}
& \chi_{i}^{n}-\sum_{g, h, a, b}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i}^{n}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b}= \\
& \quad \sum_{g, h, a, b, j, k}\left(\partial_{g h} H \partial_{j k, x y}^{2} G+\partial_{g h} G \partial_{j k, x y}^{2} H\right)\left(C_{i}^{n}\right)\left(\lambda_{i}^{n, g h}+\frac{1}{2} \nu_{i}^{n, g h}\right) \lambda_{i}^{n, a b} \lambda_{i}^{n, j k}+\varphi_{i}^{n}, \quad \text { and } \\
& \sum_{g, h, a, b}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i}^{\prime} n\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i}^{n}\right)= \\
& \sum_{g, h, a, b, x, y}\left(\partial_{g h} H \partial_{a b, x y}^{2} G+\partial_{a b} G \partial_{g h, x y}^{2} G\right)\left(C_{i}^{n}\right)\left(\nu_{i}^{n, x y}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b}+\delta_{i}^{n}
\end{aligned}
$$

with $\mathbb{E}\left(\left|\varphi_{i}^{n}\right| \mid \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}$ and $\mathbb{E}\left(\left|\delta_{i}^{n}\right| \mid \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}$ which follow by the Cauchy-Schwartz inequality together with equation (E.30). Given that $k_{n}=\theta\left(\Delta_{n}\right)^{-1 / 2}$, the previous inequalities imply

$$
\frac{3 \Delta_{n}^{-1 / 4}}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \varphi_{i}^{n} \stackrel{\mathbb{P}}{\Longrightarrow} 0 \text { and } \frac{3 \Delta_{n}^{-1 / 4}}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \delta_{i}^{n} \xlongequal{\mathbb{P}} 0
$$

Therefore, it suffices to show that

$$
\begin{align*}
& \frac{3 \Delta_{n}^{-1 / 4}}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \sum_{g, h, a, b, j, k}\left(\partial_{g h} H \partial_{j k, a b}^{2} G+\partial_{g h} H \partial_{j k, a b}^{2} G\right)\left(C_{i}^{n}\right) \lambda_{i}^{n, g h} \lambda_{i}^{n, a b} \lambda_{i}^{n, j k} \xrightarrow{\mathbb{P}} 0  \tag{E.32}\\
& \frac{3 \Delta_{n}^{-1 / 4}}{2 k_{n}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \sum_{g, h, a, b, j, k}\left(\partial_{g h} H \partial_{j k, a b}^{2} G+\partial_{g h} H \partial_{j k, a b}^{2} G\right)\left(C_{i}^{n}\right) \nu_{i}^{n, g h} \lambda_{i}^{n, a b} \lambda_{i}^{n, j k} \xrightarrow{\mathbb{P}} 0 \tag{E.33}
\end{align*}
$$

These results hold by the bounds in Lemma B5.

## E. 3 Proof of Theorem B3

First, we decompose the approximated estimator as

$$
\begin{equation*}
\left[H(\widehat{C), G}(C)]_{T}^{(A)}=\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}-\left[H(\widehat{C), G}(C)]_{T}^{(A 2)},\right.\right.\right. \tag{E.34}
\end{equation*}
$$

with

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}=\frac{3}{2 k_{n}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}^{n}\right)\left(\widehat{C}_{i+k_{n}}^{\prime n, g h}-\widehat{C}_{i}^{\prime n, g h}\right)\left(\widehat{C}_{i+k_{n}}^{\prime n, a b}-\widehat{C}_{i}^{\prime n, a b}\right),\right.
$$

and

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 2)}=\frac{3}{k_{n}^{2}} \sum_{g, h, a, b=1}^{d} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left(\partial_{g h} H \partial_{a b} G\right)\left(\widehat{C}_{i}^{\prime n}\right)\left(\widehat{C}_{i}^{\prime n, g a} \widehat{C}_{i}^{\prime n, h b}+\widehat{C}_{i}^{\prime n, g b} \widehat{C}_{i}^{\prime n, h a}\right)\right.
$$

In this section, we use the notation $C_{i-1}^{n}=C_{(i-1) \Delta_{n}}$ and $\mathcal{F}_{i}=\mathcal{F}_{(i-1) \Delta_{n}}$ to simplify the exposition. Given the boundedness of the derivatives of $H$ and $G$ and the fact that $k_{n}=\theta\left(\Delta_{n}\right)^{-1 / 2}$, by Theorem 2.2 in Jacod and Rosenbaum (2015) we have

$$
\frac{1}{\sqrt{\Delta_{n}}}\left(\left[H(\widehat{C), G}(C)]_{T}^{(A 2)}-\frac{3}{\theta^{2}} \sum_{g, h, a, b=1}^{d} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right)\left(c_{t}^{g a} c_{t}^{h b}+c_{i}^{g b} c_{t}^{h a}\right) d t\right)=O_{p}(1)\right.
$$

which yields

$$
\frac{1}{\Delta_{n}^{1 / 4}}\left(\left[H(\widehat{C), G}(C)]_{T}^{(A 2)}-\frac{3}{\theta^{2}} \sum_{g, h, a, b=1}^{d} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right)\left(c_{t}^{g a} c_{t}^{h b}+c_{i}^{g b} c_{t}^{h a}\right) d t\right) \xrightarrow{\mathbb{P}} 0\right.
$$

Using the multivariate quantities defined in Section A, we can show that the following decompositions hold:

$$
\begin{aligned}
\widehat{C}_{i}^{\prime n} & =C_{i-1}^{n}+\frac{1}{k_{n}} \sum_{j=0}^{k_{n}-1} \sum_{u=1}^{2} \bar{\varepsilon}(u)_{j}^{n} \zeta(u)_{i+j}^{n}, \quad \widehat{C}_{i+k_{n}}^{\prime n}-\widehat{C}_{i}^{\prime n}=\frac{1}{k_{n}} \sum_{j=0}^{2 k_{n}-1} \sum_{u=1}^{2} \varepsilon(u)_{j}^{n} \zeta(u)_{i+j}^{n}, \\
\lambda_{i}^{n, g h} \lambda_{i}^{n, a b} & =\frac{1}{k_{n}^{2}} \sum_{u=1}^{2} \sum_{v=1}^{2}\left(\sum_{j=0}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+j}^{n, a b}\right. \\
& \left.+\sum_{j=0}^{2 k_{n}-2} \sum_{q=j+1}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{q}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+q}^{n, a b}+\sum_{j=1}^{2 k_{n}-1} \sum_{q=0}^{j-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{q}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+q}^{n, a b}\right) .
\end{aligned}
$$

Changing the order of the summation in the last term yields

$$
\begin{aligned}
\lambda_{i}^{n, g h} \lambda_{i}^{n, a b} & =\frac{1}{k_{n}^{2}} \sum_{u=1}^{2} \sum_{v=1}^{2}\left(\sum_{j=0}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+j}^{n, a b}\right. \\
& \left.+\sum_{j=0}^{2 k_{n}-2} \sum_{q=j+1}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{q}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+q}^{n, a b}+\sum_{j=0}^{2 k_{n}-2} \sum_{q=j+1}^{2 k_{n}-1} \varepsilon(v)_{j}^{n} \varepsilon(u)_{q}^{n} \zeta(v)_{i+j}^{n, a b} \zeta(u)_{i+q}^{n, g h}\right) .
\end{aligned}
$$

Therefore, we can further rewrite $\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}\right.$ as

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 1)}=\left[H(\widehat{C), G}(C)]_{T}^{(A 11)}+\left[H(\widehat{C), G}(C)]_{T}^{(A 12)}+\left[H(\widehat{C), G}(C)]_{T}^{(A 13)},\right. \text { with }\right.\right.\right.
$$

$$
\left[H(\widehat{C), G}(C)]_{T}^{(A 1 w)}=\sum_{g, h, a, b=1}^{d} \sum_{u, v=1}^{2} \widehat{A 1 w}(H, g h, u ; G, a b, v)_{T}^{n}, \quad w=1,2,3,\right.
$$

and

$$
\begin{aligned}
& \widehat{A 11}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \sum_{j=0}^{2 k_{n}-1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+j}^{n, a b}, \\
& \widehat{A 12}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \sum_{j=0}^{2 k_{n}-2} \sum_{q=j+1}^{2 k_{n}-1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{q}^{n} \zeta(u)_{i+j}^{n, g h} \zeta(v)_{i+q}^{n, a b}, \\
& \widehat{A 13}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \sum_{j=0}^{2 k_{n}-2} \sum_{q=j+1}^{2 k_{n}-1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}^{n}\right) \varepsilon(v)_{j}^{n} \varepsilon(u)_{q}^{n} \zeta(v)_{i+j}^{n, a b} \zeta(u)_{i+q}^{n, g h},
\end{aligned}
$$

where we clearly have $\widehat{A 13}(H, g h, u ; G, a b, v)_{T}^{n}=\widehat{A 12}(G, a b, v ; H, g h, u)_{T}^{n}$. By a change of the order of the summation,

$$
\begin{aligned}
\widehat{A 11}(H, g h, u ; G, a b, v)_{T}^{n} & =\frac{3}{2 k_{n}^{3}} \sum_{i=1}^{\left[T / \Delta_{n}\right]} \sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right) \\
& \times\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n} \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}, \\
\widehat{A 12}(H, g h, u ; G, a b, v)_{T}^{n} & =\frac{3}{2 k_{n}^{3}} \sum_{i=2}^{\left[T / \Delta_{n}\right](i-1) \wedge\left(2 k_{n}-1\right)} \sum_{m=1}^{\left(2 k_{n}-m-1\right) \wedge(i-m-1)} \sum_{j=0 \vee\left(i+2 k_{n}-1-m-\left[T / \Delta_{n}\right]\right)}^{\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right)} \\
& \times \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n} \zeta_{g h}(u)_{i-m}^{n} \zeta_{a b}(v)_{i}^{n} .
\end{aligned}
$$

Now, set

$$
\begin{aligned}
& \widetilde{A 11}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \sum_{j=0}^{2 k_{n}-1}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n} \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}, \\
& \widetilde{A 12}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right](i-1) \wedge\left(2 k_{n}-1\right)} \sum_{m=1}^{\left(2 k_{n}-m-1\right)} \sum_{j=0}^{n}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1-m}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n} \\
& \quad \times \zeta_{g h}(u)_{i-m}^{n} \zeta_{a b}(v)_{i}^{n},
\end{aligned}
$$

and

$$
\begin{aligned}
& \overline{A 11}(H, g h, u ; G, a b, v)_{T}^{n}=\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\sum_{j=0}^{2 k_{n}-1} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right)\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \\
& =\lambda(u, v)_{0}^{n} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b},
\end{aligned}
$$

$\overline{A 12}(H, g h, u ; G, a b, v)_{T}^{n}$

$$
\begin{aligned}
& =\frac{3}{2 k_{n}^{3}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \sum_{m=1}^{(i-1) \wedge\left(2 k_{n}-1\right)} \sum_{j=0}^{\left(2 k_{n}-m-1\right)} \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n} \zeta_{g h}(u)_{i-m}^{n} \zeta_{a b}(v)_{i}^{n} \\
& =\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \rho_{g h}(u, v)_{i}^{n} \zeta_{a b}(v)_{i}^{n},
\end{aligned}
$$

with

$$
\rho_{g h}(u, v)_{i}^{n}=\sum_{m=1}^{2 k_{n}-1} \lambda(u, v)_{m}^{n} \zeta_{g h}(u)_{i-m}^{n}
$$

We show below that the following results hold:

$$
\begin{align*}
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\widehat{A 1 w}(H, g h, u ; G, a b, v)_{T}^{n}-\widetilde{A 1 w}(H, g h, u ; G, a b, v)_{T}^{n}\right) \xrightarrow{\mathbb{P}} 0  \tag{E.35}\\
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\widetilde{A 1 w}(H, g h, u ; G, a b, v)_{T}^{n}-\widetilde{A 1 w}(H, g h, u ; G, a b, v)_{T}^{n}\right) \xrightarrow{\mathbb{P}} 0 \tag{E.36}
\end{align*}
$$

for all $(H, g h, u, G, a b, v)$ and $w=1,2$.

## E.3.1 Proof of Equation (E.35) for $w=1$

To prove this result, first, notice that the $\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}$ are scaled by random variables rather that constant real numbers. Next, observe that we can write

$$
\begin{aligned}
& \widehat{A 11}-\widetilde{A 11}=\widetilde{\widehat{A 11}}(1)+\widetilde{\widehat{A 11}}(2)+\widetilde{\widehat{A 11}}(3) \text { with } \\
& \widetilde{\widehat{A 11}}(1)= \sum_{i=1}^{\left(2 k_{n}-1\right) \wedge\left[T / \Delta_{n}\right]}\left(\frac{3}{2 k_{n}^{3}} \sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}, \\
& \widetilde{\widehat{A 11}}(2)=\sum_{i=\left[T / \Delta_{n}\right]-2 k_{n}+2}^{\left[T / \Delta_{n}\right]} \frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right. \\
&\left.-\sum_{j=0}^{\left(2 k_{n}-1\right)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}, \\
& \widetilde{\widehat{A 11}}(3)= \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right. \\
&\left.-\sum_{j=0}^{\left(2 k_{n}-1\right)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} .
\end{aligned}
$$

It is easy to see that $\widehat{\widehat{A 12}}(3)=0$. Using equation (B.10) with $Z=c$ and equation (E.29), we obtain

$$
\begin{equation*}
\mathbb{E}\left(\left\|\zeta(1)_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K_{q}, \quad \mathbb{E}\left(\left\|\zeta(2)_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K_{q} \Delta_{n}^{q / 2} \tag{E.37}
\end{equation*}
$$

By the boundedness of the derivatives of $H$ and $G$, the random quantities $\left(\frac{3}{2 k_{n}^{3}} \sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right)$ and
$\frac{3}{2 k_{n}^{3}} \sum_{j=0}^{\left(2 k_{n}-1\right)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}$ are $\mathcal{F}_{i-1}^{n}-$ measurable and are bounded by $\widetilde{\lambda}_{u, v}^{n}$ defined as

$$
\widetilde{\lambda}_{u, v}^{n}= \begin{cases}K & \text { if }(u, v)=(2,2) \\ K / k_{n} & \text { if }(u, v)=(1,2),(2,1) \\ K / k_{n}^{2} & \text { if }(u, v)=(1,1)\end{cases}
$$

Similarly, the quantity

$$
\frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-1\right) \wedge(i-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}-\sum_{j=0}^{\left(2 k_{n}-1\right)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}\right)
$$

is $\mathcal{F}_{i-1}^{n}-$ measurable and bounded by $2 \widetilde{\lambda}_{u, v}^{n}$. Note also that, by equation (E.37) and the Cauchy Schwartz inequality, we have

$$
\begin{aligned}
& \mathbb{E}\left(\left|\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}\right| \mid \mathcal{F}_{i-1}^{n}\right) \leq \mathbb{E}\left(\left\|\zeta(u)_{i}^{n}\right\|^{2} \mid \mathcal{F}_{i-1}^{n}\right)^{1 / 2} \mathbb{E}\left(\left\|\zeta(v)_{i}^{n}\right\|^{2} \mid \mathcal{F}_{i-1}^{n}\right)^{1 / 2} \\
& \leq \leq \begin{array}{ll}
K \Delta_{n} & \text { if }(u, v)=(2,2) \\
K \Delta_{n}^{1 / 2} & \text { if }(u, v)=(1,2),(2,1) \\
K & \text { if }(u, v)=(1,1)
\end{array}
\end{aligned}
$$

The above bounds, together with the fact that $k_{n}=\theta \Delta_{n}^{-1 / 2}$, imply $\mathbb{E}(|\widetilde{\widehat{A 11}}(1)|) \leq K \Delta_{n}^{1 / 2}$ and $\mathbb{E}(|\widetilde{\widehat{A 11}}(2)|) \leq$ $K \Delta_{n}^{1 / 2}$ for all $(u, v)$. These two results together imply $\widetilde{\widehat{A 11}}(1)=o\left(\Delta_{n}^{-1 / 4}\right)$ and $\widetilde{\widehat{A 11}}(2)=o\left(\Delta_{n}^{-1 / 4}\right)$, which yields the result.

## E.3.2 Proof of Equation (E.35) for $w=2$

First, observe that $\widehat{A 12}-\widetilde{A 12}=\widetilde{\widehat{A 12}}(1)+\widetilde{\widehat{A 12}}(2)$, with

$$
\begin{aligned}
\widetilde{A 12}(1)= & \sum_{i=2}^{\left(2 k_{n}-1\right) \wedge\left[T / \Delta_{n}\right]}\left(\sum_{m=1}^{(i-1)} \frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-m-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-m-1\right) \wedge(i-m-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n}\right)\right. \\
& \left.\times \zeta_{g h}(u)_{i-m}^{n}\right) \zeta_{a b}(v)_{i}^{n}, \\
\widetilde{\widehat{A 12}}(2)= & \sum_{i=\left[T / \Delta_{n}\right]-2 k_{n}+2}^{\left[T / \Delta_{n}\right]}\left(\sum _ { m = 1 } ^ { ( i - 1 ) \wedge ( 2 k _ { n } - 1 ) } \left(\frac{3}{2 k_{n}^{3}} \sum_{j=0 \vee\left(i+2 k_{n}-1-m-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-m-1\right) \wedge(i-m-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right) \varepsilon(u)_{j}^{n}\right.\right. \\
& \left.\left.\left.\times \varepsilon(v)_{j+m}^{n}\right)-\sum_{j=0}^{\left(2 k_{n}-m-1\right)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n}\right) \zeta_{g h}(u)_{i-m}^{n}\right) \zeta_{a b}(v)_{i}^{n} .
\end{aligned}
$$

Notice that the quantity

$$
\kappa_{i}^{m, n}=\frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-m-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-m-1\right) \wedge(i-m-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n}\right)
$$

is $\mathcal{F}_{i-m-1}^{n}$ measurable and bounded by $\widetilde{\lambda}_{u, v}^{n}$. Let

$$
\kappa_{i}^{n}=\sum_{m=1}^{(i-1)} \frac{3}{2 k_{n}^{3}}\left(\sum_{j=0 \vee\left(i+2 k_{n}-1-m-\left[T / \Delta_{n}\right]\right)}^{\left(2 k_{n}-m-1\right) \wedge(i-m-1)}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1-j-m}^{n}\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n}\right) \zeta_{g h}(u)_{i-m}^{n}
$$

It follows that $\kappa_{i}^{n}$ is $\mathcal{F}_{i-1}^{n}$-measurable and we have

$$
\begin{aligned}
\mathbb{E}\left(\left|\kappa_{i}^{m, n}\right|^{z} \mid \mathcal{F}_{0}\right) & \leq\left(\widetilde{\lambda}_{u, v}^{n}\right)^{z}, \\
\left|\mathbb{E}\left(\zeta(u)_{i-m}^{n} \mid \mathcal{F}_{i-m-1}\right)\right| & \leq \begin{cases}K \sqrt{\Delta_{n}} & \text { if } u=1 \\
K \Delta_{n} & \text { if } u=2\end{cases}
\end{aligned}
$$

$$
\mathbb{E}\left(\left\|\zeta(u)_{i-m}^{n}\right\|^{z} \mid \mathcal{F}_{i-m-1}\right) \leq \begin{cases}K_{z} & \text { if } u=1 \\ K_{z} \Delta_{n}^{z / 2} & \text { if } u=2\end{cases}
$$

Using Lemma B3, we deduce that for $z \geq 2$,

$$
\mathbb{E}\left(\left|\kappa_{i}^{n}\right|^{z}\right) \leq\left\{\begin{array}{ll}
K_{z}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{z} k_{n}^{z / 2} & \text { if } u=1 \\
K_{z}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{z} / k_{n}^{z / 2} & \text { if } u=2
\end{array} \leq \begin{cases}K_{z} / k_{n}^{-3 z / 2} & \text { if } v=1 \\
K_{z} k_{n}^{-z / 2} & \text { if } v=2\end{cases}\right.
$$

Using the above result, we obtain $\frac{1}{\Delta_{n}^{1 / 4}} \widehat{\widehat{A 12}}(1) \stackrel{\mathbb{P}}{\Rightarrow} 0$. A similar argument yields $\frac{1}{\Delta_{n}^{1 / 4}} \widehat{\widehat{A 12}}(2) \stackrel{\mathbb{P}}{\Rightarrow} 0$, which completes the proof of the equation (E.35) for $w=2$.

## E.3.3 Proof of Equation (E.36) for $w=1$

Define

$$
\Theta(u, v)_{0}^{(C), i, n}=\frac{3}{2 k_{n}^{3}} \sum_{j=0}^{2 k_{n}-1}\left(\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right)\right) \varepsilon(u)_{j}^{n} \varepsilon(v)_{j}^{n}
$$

By Taylor expansion, boundedness of the derivatives of $H$ and $G$, and using (B.10) with $Z=c$, we have

$$
\begin{aligned}
& \left|\mathbb{E}\left(\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right) \mid \mathcal{F}_{i-2 k_{n}}^{n}\right)\right| \leq K\left(k_{n} \Delta_{n}\right) \leq K \sqrt{\Delta_{n}} \\
& \mathbb{E}\left(\left|\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-j-1}^{n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}^{n}\right)\right|^{q} \mid \mathcal{F}_{i-2 k_{n}}^{n}\right) \mid \leq K\left(k_{n} \Delta_{n}\right)^{q / 2} \leq K \Delta_{n}^{q / 4},
\end{aligned}
$$

for $q \geq 2$ and for $j=0, \ldots, 2 k_{n}-1$. Next, observe that $\Theta(u, v)_{0}^{(C), i, n}$ is $\mathcal{F}_{i-1}^{n}$-measurable and satisfies $\left|\Theta(u, v)_{0}^{(C), i, n}\right| \leq \widetilde{\lambda}_{u, v}^{n},\left|\mathbb{E}\left(\Theta(u, v)_{0}^{(C), i, n} \mid \mathcal{F}_{i-2 k_{n}}^{n}\right)\right| \leq K \Delta_{n}^{1 / 2} \widetilde{\lambda}_{u, v}^{n}$ and $\mathbb{E}\left(\left|\Theta(u, v)_{0}^{(C), i, n}\right| q \mid \mathcal{F}_{i-2 k_{n}}^{n}\right) \leq$ $K_{q} \Delta_{n}^{q / 4}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{q}$ where the latter follows from the Hölder inequality. We aim to prove that

$$
\widehat{E}=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n} \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}\right]
$$

converges to zero in probability for any $H, G, g, h, a$, and $b$ with $u, v=1,2$. To show this result, we first introduce the following quantities:

$$
\begin{aligned}
& \widehat{E}(1)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n} \mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)\right] \\
& \widehat{E}(2)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}-\mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)\right)\right]
\end{aligned}
$$

with $\widehat{E}=\widehat{E}(1)+\widehat{E}(2)$. By Cauchy-Schwartz inequality, we have

$$
\mathbb{E}\left(\left|\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}\right|^{q}\right) \leq\left(\widehat{\lambda}_{u, v}^{n}\right)^{q / 2}, \text { where } \widehat{\lambda}_{u, v}^{n}= \begin{cases}K & \text { if }(u, v)=(1,1) \\ K \Delta_{n} & \text { if }(u, v)=(1,2),(2,1) \\ K \Delta_{n}^{2} & \text { if }(u, v)=(2,2)\end{cases}
$$

Since $\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}$ is $\mathcal{F}_{i}^{n}$-measurable,
the martingale property of $\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}-\mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)$ implies, for all $(u, v)$,

$$
\mathbb{E}\left(|\widehat{E}(2)|^{2}\right) \leq K \Delta_{n}^{-3 / 2}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n}\right)^{2} \widehat{\lambda}_{u, v}^{n} \leq K \Delta_{n}
$$

The latter inequality implies $\widehat{E}(2) \stackrel{\mathbb{P}}{\Rightarrow} 0$ for all $(u, v)$. It remains to show that $\widehat{E}(1) \stackrel{\mathbb{P}}{\Rightarrow} 0$. Here, we recall some bounds under Assumption 2,

$$
\begin{align*}
& \left|\mathbb{E}\left(\zeta(1)_{i}^{n, g h} \zeta(2)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)\right| \leq K \Delta_{n},  \tag{E.38}\\
& \left|\mathbb{E}\left(\zeta(1)_{i}^{n, g h} \zeta(1)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)-\left(C_{i-1}^{n, g a} C_{i-1}^{n, h b}+C_{i-1}^{n, g b} C_{i-1}^{n, h a}\right)\right| \leq K \Delta_{n}^{1 / 2},  \tag{E.39}\\
& \left|\mathbb{E}\left(\zeta(2)_{i}^{n, g h} \zeta(2)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}-\bar{C}_{i-1}^{n, g h, a b} \Delta_{n}\right)\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i}^{n}\right) \tag{E.40}
\end{align*}
$$

Case $(u, v) \in\{(1,2),(2,1)\}$. By equation (E.38) we have

$$
\mathbb{E}(|\widehat{E}(1)|) \leq K \frac{T}{\Delta_{n}} \frac{1}{\Delta_{n}^{1 / 4}}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n} \Delta_{n}\right) \leq K \Delta_{n}^{1 / 2} \quad \text { so } \quad \widehat{E}(1) \stackrel{\mathbb{P}}{\Rightarrow} 0
$$

Case $(u, v) \in\{(1,1),(2,2)\}$. Set

$$
\begin{aligned}
& \widehat{E}^{\prime}(1)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n} V_{i-2 k_{n}}^{n}\right] \\
& \widehat{E}^{\prime \prime}(1)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n}\left(V_{i-1}^{n}-V_{i-2 k_{n}}^{n}\right)\right] \\
& \widehat{E}^{\prime \prime \prime}(1)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \Theta(u, v)_{0}^{(C), i, n}\left(\mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)-V_{i-1}^{n}\right)\right]
\end{aligned}
$$

where

$$
V_{i-1}^{n}= \begin{cases}C_{i-1}^{n, g a} C_{i-1}^{n, h b}+C_{i-1}^{n, g b} C_{i-1}^{n, h a} & \text { if }(u, v)=(2,2) \\ \bar{C}_{i-1}^{n, g h, a b} \Delta_{n} & \text { if }(u, v)=(1,1) \\ 0 & \text { otherwise }\end{cases}
$$

Note that we have $\widehat{E}(1)=\widehat{E}^{\prime}(1)+\widehat{E}^{\prime \prime}(1)+\widehat{E}^{\prime \prime \prime}(1)$. Using equations (E.39) and (E.40), it can be shown that

$$
\mathbb{E}\left(\left|\widehat{E}^{\prime \prime \prime}(1)\right|\right) \leq\left\{\begin{array}{ll}
K \frac{1}{\Delta_{n}^{5 / 4}}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n}\right) \Delta_{n}^{1 / 2} & \text { if }(u, v)=(1,1) \\
K \frac{1}{\Delta_{n}^{5 / 4}}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n}\right) \Delta_{n}^{3 / 2} & \text { if }(u, v)=(2,2)
\end{array} \leq K \Delta_{n}^{1 / 2} \quad\right. \text { in all cases. }
$$

Next, we prove $\widehat{E}^{\prime}(1) \stackrel{\mathbb{P}}{\Rightarrow} 0$. To this end, write

$$
\widehat{E}^{\prime}(1)=\frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \Theta(u, v)_{0}^{(C), i-1+2 k_{n}, n} V_{(i-1) \Delta_{n}}\right]
$$

Using the $\mathcal{F}_{i+2 k_{n}-2}^{n}$-measurability of the last sum, we are able to show

$$
\begin{aligned}
& \frac{1}{\Delta_{n}^{1 / 4}}\left[\sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1}\left|\mathbb{E}\left(\Theta(u, v)_{0}^{(C), i-1+2 k_{n}, n} V_{(i-1) \Delta_{n}} \mid \mathcal{F}_{i-1}^{n}\right)\right|\right] \stackrel{\mathbb{P}}{\Rightarrow} 0 \quad \text { and } \\
& \left.\frac{2 k_{n}-2}{\Delta_{n}^{1 / 2}}\left[\left.\sum_{i=1}^{\left[T / \Delta_{n}\right]-2 k_{n}+1} \mathbb{E}\left(\mid \Theta(u, v)_{0}^{(C), i-1+2 k_{n}, n} V_{(i-1) \Delta_{n}}\right)\right|^{2}\right)\right] \Rightarrow 0
\end{aligned}
$$

The first result readily follows from the inequality

$$
\left|\mathbb{E}\left(\Theta(u, v)_{0}^{(C), i-1+2 k_{n}, n} V_{(i-1) \Delta_{n}} \mid \mathcal{F}_{i-1}^{n}\right)\right| \leq\left\{\begin{array}{ll}
K \Delta_{n}^{1 / 2} \widetilde{\lambda}_{u, v}^{n} & \text { if }(u, v)=(1,1) \\
K \Delta_{n}^{1 / 2} \widetilde{\lambda}_{u, v}^{n} \Delta_{n} & \text { if }(u, v)=(2,2)
\end{array} \leq K \Delta_{n}^{3 / 2} \quad\right. \text { in all cases }
$$

while the second is a direct consequence of

$$
\mathbb{E}\left(\left|\Theta(u, v)_{0}^{(C), i-1+2 k_{n}, n} V_{(i-1) \Delta_{n}}\right|^{2}\right) \leq\left\{\begin{array}{ll}
K \Delta_{n}^{1 / 2}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{2} & \text { if }(u, v)=(1,1) \\
K \Delta_{n}^{1 / 2}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{2} \Delta_{n}^{2} & \text { if }(u, v)=(2,2)
\end{array} \leq K \Delta_{n}^{5 / 2} \quad\right. \text { in all cases. }
$$

Finally, to prove that $\widehat{E}^{\prime \prime}(1) \stackrel{\mathbb{P}}{\Longrightarrow} 0$, we use the fact that

$$
\begin{aligned}
\mathbb{E}\left(\left|\Theta(u, v)_{0}^{(C), i, n}\left(V_{(i-1) \Delta_{n}}-V_{\left(i-2 k_{n}\right) \Delta_{n}}\right)\right|\right) & \leq \mathbb{E}\left(\left|\Theta(u, v)_{0}^{(C), i, n}\right|^{2}\right)^{1 / 2} \mathbb{E}\left(\left|V_{(i-1) \Delta_{n}}-V_{\left(i-2 k_{n}\right) \Delta_{n}}\right|^{2}\right)^{1 / 2} \\
& \leq \begin{cases}K \Delta_{n}^{1 / 2} \widetilde{\lambda}_{u, v}^{n} & \text { if }(u, v)=(1,1) \\
K \Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n} \Delta_{n} \Delta_{n}^{1 / 4} & \text { if }(u, v)=(2,2)\end{cases}
\end{aligned}
$$

which follows from the Cauchy-Schwartz inequality and earlier bounds. In particular, successive conditioning together with Assumption 2 imply that for $(u, v)=(1,1)$ and $(2,2)$,
$\mathbb{E}\left(\left|V_{(i-1) \Delta_{n}}-V_{\left(i-2 k_{n}\right) \Delta_{n}}\right|^{2}\right) \leq \Delta_{n}^{1 / 2}$.

## E.3.4 Proof of Equation (E.36) for $w=2$

Our aim here is to show that
$\widehat{E}(2)=\frac{1}{\Delta_{n}^{1 / 4}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\sum_{m=1}^{2 k_{n}-1}\left(\frac{3}{2 k_{n}^{3}} \sum_{j=0}^{2 k_{n}-m-1}\left[\left(\partial_{g h} H \partial_{a b} G\right)\left(c_{i-j-m-1}^{n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(c_{i-2 k_{n}}^{n}\right)\right] \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n}\right) \times\right.$ $\left.\zeta(u)_{i-m}^{n, g h}\right) \zeta(v)_{i}^{n, a b} \stackrel{\mathbb{P}}{\Longrightarrow} 0$.

For this purpose, we introduce some new notation. For any $0 \leq m \leq 2 k_{n}-1$, set

$$
\begin{aligned}
& \Theta(u, v)_{m}^{(C), i, n}=\frac{3}{2 k_{n}^{3}} \sum_{j=0}^{2 k_{n}-m-1}\left[\left(\partial_{g h} H \partial_{a b} G\right)\left(c_{i-j-m-1}^{n}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(c_{i-2 k_{n}}^{n}\right)\right] \varepsilon(u)_{j}^{n} \varepsilon(v)_{j+m}^{n} \\
& \rho(u, v)^{(C), i, n, g h}=\sum_{m=1}^{2 k_{n}-1} \Theta(u, v)_{m}^{(C), i, n} \zeta(u)_{i-m}^{n, g h}
\end{aligned}
$$

It is easy to see that $\Theta(u, v)_{m}^{(C), i, n}$ is $\mathcal{F}_{i-m-1}^{n}$ measurable and satisfies, by Hölder inequality,

$$
\left|\Theta(u, v)_{m}^{(C), i, n}\right| \leq \widetilde{\lambda}_{u, v}^{n} \text { and } \mathbb{E}\left(\left|\Theta(u, v)_{m}^{(C), i, n}\right| q \mid \mathcal{F}_{i-2 k_{n}}^{n}\right) \leq K_{q} \Delta_{n}^{q / 4}\left(\widetilde{\lambda}_{u, v}^{n}\right)^{q}
$$

Lemma B3 implies that for $q \geq 2$,

$$
\mathbb{E}\left(\left|\rho(u, v)^{(C), i, n, g h}\right|^{q}\right) \leq\left\{\begin{array}{ll}
K_{q}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n}\right)^{q} k_{n}^{q / 2} & \text { if } u=1  \tag{E.41}\\
K_{q}\left(\Delta_{n}^{1 / 4} \widetilde{\lambda}_{u, v}^{n}\right)^{q} / k_{n}^{q / 2} & \text { if } u=2
\end{array} \leq \begin{cases}K_{q} / k_{n}^{2 q} & \text { if } v=1 \\
K_{q} k_{n}^{q} & \text { if } v=2\end{cases}\right.
$$

Set

$$
\begin{aligned}
& \widehat{E}^{\prime}(2)=\frac{1}{\Delta_{n}^{1 / 4}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \rho(u, v)^{(C), i, n, g h} \mathbb{E}\left(\zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right), \\
& \widehat{E}^{\prime \prime}(2)=\frac{1}{\Delta_{n}^{1 / 4}} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]} \rho(u, v)^{(C), i, n, g h}\left(\zeta(v)_{i}^{n, a b}-\mathbb{E}\left(\zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)\right) .
\end{aligned}
$$

The martingale increments property implies $\mathbb{E}\left(\left|\widehat{E}^{\prime \prime}(2)\right|^{2}\right) \leq K \Delta_{n}^{1 / 2}$ in all the cases, which in turn implies $\widehat{E}^{\prime \prime}(2) \stackrel{\mathbb{P}}{\Longrightarrow} 0$. Next, using the bounds on $\rho(u, v)^{(C), i, n, g h}$, we obtain that $\widehat{E}^{\prime}(2) \xrightarrow{\mathbb{P}} 0$.

We refer to Jacod and Rosenbaum (2015) for the proofs of Lemma B1 and Lemma B2.

## E. 4 Proof of Lemma B3

Set

$$
\xi_{i}^{n}=\varphi_{i-1}^{n} \zeta_{i}^{n}, \quad \xi_{i}^{\prime n}=\mathbb{E}\left(\xi_{i} \mid \mathcal{F}_{i-1}^{n}\right)=\mathbb{E}\left(\varphi_{i-1}^{n} \zeta_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right)=\varphi_{i-1}^{n} \mathbb{E}\left(\zeta_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right), \quad \text { and } \quad \xi_{i}^{\prime \prime n}=\xi_{i}^{n}-\xi_{i}^{\prime n}
$$

Given that $\left\|\mathbb{E}\left(\zeta_{i}^{n} \mid \mathcal{F}_{i-1}^{n}\right)\right\| \leq L^{\prime}$, we have $\left\|\xi_{i}^{\prime n}\right\| \leq L^{\prime}\left|\varphi_{i-1}^{n}\right|$. By the convexity of the function $x^{q}$, which holds for $q \geq 2$, we have

$$
\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{n}\right\|^{q} \leq K\left(\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{\prime} n\right\|^{q}+\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{\prime \prime}\right\|^{q}\right) .
$$

Therefore, on the one hand we have

$$
\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{\prime n}\right\|^{q} \leq K k_{n}^{q-1} \sum_{j=1}^{2 k_{n}-1}\left\|\xi_{i+j}^{\prime n}\right\|^{q} \leq K k_{n}^{q-1} L^{\prime q} \sum_{j=1}^{2 k_{n}-1}\left|\varphi_{i+j-1}^{n}\right|^{q}
$$

which by $\mathbb{E}\left(\left\|\varphi_{i+j-1}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L^{q}$, satisfies

$$
\mathbb{E}\left(\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{\prime n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K L^{\prime q} k_{n}^{q-1} \sum_{j=1}^{2 k_{n}-1} \mathbb{E}\left(\left|\varphi_{i+j-1}^{n}\right|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K L^{\prime q} k_{n}^{q} L^{q}
$$

On the other hand, we have $\mathbb{E}\left(\left\|\xi_{i+j}^{\prime \prime n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq \mathbb{E}\left(\left\|\xi_{i+j}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L_{q} L^{q}$ and $\mathbb{E}\left(\xi_{i+j}^{\prime \prime n} \mid \mathcal{F}_{i-1}^{n}\right)=0$, where the first inequality is a consequence of $\mathbb{E}\left(\left\|\xi_{i+j}^{\prime n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq \mathbb{E}\left(\left\|\xi_{i+j}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L_{q} L^{q}$, which follows from the Jensen's inequality and the law of iterated expectation. Hence, by Lemma B. 2 of Aït-Sahalia and Jacod (2014) we have

$$
\mathbb{E}\left(\left\|\sum_{j=1}^{2 k_{n}-1} \xi_{i+j}^{\prime \prime}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq K_{q} L^{q} L_{q} k_{n}^{q / 2}
$$

To see the latter, we first prove that the required condition $\left.\mathbb{E}\left(\left\|\xi_{i}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L_{q} L^{q}\right)$ in the Lemma B. 2 of Aït-Sahalia and Jacod (2014) can be replaced by $\left.\mathbb{E}\left(\left\|\xi_{i+j}^{n}\right\|^{q} \mid \mathcal{F}_{i-1}^{n}\right) \leq L_{q} L^{q}\right)$ for $1 \leq j \leq 2 k_{n}-1$ without altering the result.

## E. 5 Proof of Lemma B4

We use the terminology "successive conditioning" to refer to either of the following two equalities,

$$
\begin{aligned}
x_{1} y_{1}-x_{0} y_{0}= & x_{0}\left(y_{1}-y_{0}\right)+y_{0}\left(x_{1}-x_{0}\right)+\left(x_{1}-x_{0}\right)\left(y_{1}-y_{0}\right) \\
x_{1} y_{1} z_{1}-x_{0} y_{0} z_{0}= & x_{0} y_{0}\left(z_{1}-z_{0}\right)+x_{0} z_{0}\left(y_{1}-y_{0}\right)+y_{0} z_{0}\left(x_{1}-x_{0}\right)+x_{0}\left(y_{0}-y_{1}\right)\left(z_{0}-z_{1}\right) \\
& +y_{0}\left(x_{0}-x_{1}\right)\left(z_{0}-z_{1}\right)+z_{0}\left(x_{0}-x_{1}\right)\left(y_{0}-y_{1}\right)+\left(x_{1}-x_{0}\right)\left(y_{1}-y_{0}\right)\left(z_{1}-z_{0}\right)
\end{aligned}
$$

which hold for any real numbers $x_{0}, y_{0}, z_{0}, x_{1}, y_{1}$, and $z_{1}$.
To prove Lemma B4, we first note that $\lambda_{i}^{n, j k} \lambda_{i}^{n, l m}$ is $\mathcal{F}_{i+2 k_{n}}^{n}$-measurable. Therefore, by the law of iterated
expectations, we have

$$
\mathbb{E}\left(\lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \lambda_{i+2 k_{n}}^{n, g h} \lambda_{i+2 k_{n}}^{n, a b} \mid \mathcal{F}_{i}^{n}\right)=\mathbb{E}\left(\lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \mathbb{E}\left(\lambda_{i+2 k_{n}}^{n, g h} \lambda_{i+2 k_{n}}^{n, a b} \mid \mathcal{F}_{i+2 k_{n}}^{n}\right) \mid \mathcal{F}_{i}^{n}\right) .
$$

By equation (3.27) in Jacod and Rosenbaum (2015), we have

$$
\begin{aligned}
& \left|\mathbb{E}\left(\lambda_{i+2 k_{n}}^{n, g h} \lambda_{i+2 k_{n}}^{n, a b} \mid \mathcal{F}_{i+2 k_{n}}^{n}\right)-\frac{2}{k_{n}}\left(C_{i+2 k_{n}}^{n, g a} C_{i+2 k_{n}}^{n, h b}+C_{i+2 k_{n}}^{n, g b} C_{i+2 k_{n}}^{n, h a}\right)-\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i+2 k_{n}}^{n, g h, a b}\right| \\
& \quad \leq K \sqrt{\Delta_{n}}\left(\Delta_{n}^{1 / 8}+\eta_{i+2 k_{n}, 2 k_{n}}^{n}\right), \text { and } \\
& \left|\mathbb{E}\left(\lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)-\frac{2}{k_{n}}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)-\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i}^{n, j k, l m}\right| \leq K \sqrt{\Delta_{n}}\left(\Delta_{n}^{1 / 8}+\eta_{i, 2 k_{n}}^{n}\right) .
\end{aligned}
$$

From the above, it follows that

$$
\begin{aligned}
& \left|\mathbb{E}\left(\left.\lambda_{i}^{n, j k} \lambda_{i}^{n, l m}\left[\mathbb{E}\left(\lambda_{i+2 k_{n}}^{n, g h} \lambda_{i+2 k_{n}}^{n, a b} \mid \mathcal{F}_{i+2 k_{n}}^{n}\right)-\frac{2}{k_{n}}\left(C_{i+2 k_{n}}^{n, g a} C_{i+2 k_{n}}^{n, h b}+C_{i+2 k_{n}}^{n, g b} C_{i+2 k_{n}}^{n, h a}\right)-\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i+2 k_{n}}^{n, g h, a b}\right] \right\rvert\, \mathcal{F}_{i}^{n}\right)\right| \\
& \quad \leq \sqrt{\Delta_{n}} \mathbb{E}\left(\left|\lambda_{i}^{n, j k}\right|\left|\lambda_{i}^{n, l m}\right|\left(\Delta_{n}^{1 / 8}+\eta_{i+2 k_{n}, 2 k_{n}}^{n}\right)| | \mathcal{F}_{i}^{n}\right) \leq K \sqrt{\Delta_{n}} \Delta_{n}^{1 / 8} \mathbb{E}\left(\left|\lambda_{i}^{n, j k}\right|\left|\lambda_{i}^{n, l m}\right| \mid \mathcal{F}_{i}^{n}\right) \\
& \quad+K \sqrt{\Delta_{n}} \mathbb{E}\left(\left|\lambda_{i}^{n, j k}\right|\left|\lambda_{i}^{n, l m}\right| \eta_{i+2 k_{n}, 2 k_{n}}^{n}| | \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}\left(\Delta_{n}^{1 / 8}+\eta_{i, 4 k_{n}}^{n}\right)
\end{aligned}
$$

where the last inequality follows from Lemma B1.
Now, using equation (B.10) successively with $Z=c$ and $Z=\bar{C}$ (recall that the latter holds under Assumption 2), together with the successive conditioning, we also have

$$
\begin{aligned}
& \left\lvert\, \mathbb{E}\left(\lambda _ { i } ^ { n , j k } \lambda _ { i } ^ { n , l m } \left[\frac{2}{k_{n}}\left(C_{i+2 k_{n}}^{n, g a} C_{i+2 k_{n}}^{n, h b}+C_{i+2 k_{n}}^{n, g b} C_{i+2 k_{n}}^{n, h a}\right)+\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i+2 k_{n}}^{n, g h, a b}-\frac{2}{k_{n}}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right)\right.\right.\right. \\
& \left.\left.\quad-\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i}^{n, g h, a b}\right] \mid \mathcal{F}_{i}^{n}\right) \mid \leq K \Delta_{n} \Delta_{n}^{1 / 4}, \\
& \left\lvert\, \mathbb{E}\left(\lambda _ { i } ^ { n , j k } \lambda _ { i } ^ { n , l m } \left[\frac{2}{k_{n}}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right)\right.\right.\right. \\
& \left.\quad+\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i}^{n, g h, a b}\right]-\left[\frac{2}{k_{n}}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)+\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i}^{n, j k, l m}\right] \\
& \left.\left.\quad \times\left[\frac{2}{k_{n}}\left(C_{i}^{n, g a} C_{i}^{n, h b}+C_{i}^{n, g b} C_{i}^{n, h a}\right)+\frac{2 k_{n} \Delta_{n}}{3} \bar{C}_{i}^{n, g h, a b}\right] \right\rvert\, \mathcal{F}_{i}^{n}\right) \mid \leq K \Delta_{n}\left(\Delta_{n}^{1 / 8}+\eta_{i, 2 k_{n}}^{n}\right) .
\end{aligned}
$$

The result derives from the last inequality.

## E.5.1 Proof of Equation (B.12) in Lemma B5

We start by obtaining some useful bounds for some important quantities. First, using the second statement in Lemma B2 applied to $Z=Y^{\prime}$, we have

$$
\begin{equation*}
\left|\mathbb{E}\left(\alpha_{i}^{n, j k} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, 1}^{n}\right) \tag{E.42}
\end{equation*}
$$

Second, by repeated application of the Cauchy-Schwartz inequality and making use of the third and last statements in Lemma B2 as well as equation (B.10) with $Z=c$, it can be shown that

$$
\begin{equation*}
\left|\mathbb{E}\left(\alpha_{i}^{n, j k} \alpha_{i}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)-\Delta_{n}^{2}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)\right| \leq K \Delta_{n}^{5 / 2} \tag{E.43}
\end{equation*}
$$

Next, by successive conditioning and using the bound in equation (B.10) for $Z=c$ as well as equations (E.42) and (E.43), we have for $0 \leq u \leq k_{n}-1$,

$$
\begin{gather*}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, u}^{n}\right)  \tag{E.44}\\
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k} \alpha_{i+u}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)-\Delta_{n}^{2}\left(C_{i}^{n, j l} C_{i}^{n, k m}+C_{i}^{n, j m} C_{i}^{n, k l}\right)\right| \leq K \Delta_{n}^{5 / 2} \tag{E.45}
\end{gather*}
$$

To show equation (B.12), we first observe that $\nu_{i}^{n, j k} \nu_{i}^{n, l m} \nu_{i}^{n, g h}$ can be decomposed as

$$
\begin{aligned}
& \nu_{i}^{n, j k} \nu_{i}^{n, l m} \nu_{i}^{n, g h}=\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, u}^{n, l m} \zeta_{i, u}^{n, g h}+\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-2} \sum_{v=u+1}^{k_{n}-1}\left[\zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m} \zeta_{i, v}^{n, g h}+\zeta_{i, u}^{n, g h} \zeta_{i, v}^{n, j k} \zeta_{i, v}^{n, l m}\right. \\
& \left.\quad+\zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, g h} \zeta_{i, v}^{n, j k}\right]+\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-2} \sum_{v=u+1}^{k_{n}-1}\left[\zeta_{i, u}^{n, j k} \zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, g h}+\zeta_{i, u}^{n, g h} \zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m}+\zeta_{i, u}^{n, l m} \zeta_{i, u}^{n, g h} \zeta_{i, v}^{n, j k}\right] \\
& \quad+\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-3} \sum_{v=u+1}^{k_{n}-2} \sum_{w=v+1}^{k_{n}-1}\left[\zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m} \zeta_{i, w}^{n, g h}+\zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, g h} \zeta_{i, w}^{n, l m}+\zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, j k} \zeta_{i, w}^{n, g h}+\zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, g h} \zeta_{i, w}^{n, j k}\right. \\
& \left.\quad+\zeta_{i, u}^{n, g h} \zeta_{i, v}^{n, l m} \zeta_{i, w}^{n, j k}+\zeta_{i, u}^{n, g h} \zeta_{i, v}^{n, j k} \zeta_{i, w}^{n, l m}\right]
\end{aligned}
$$

with $\zeta_{i, u}^{n}=\alpha_{i+u}^{n}+\left(C_{i+u}^{n}-C_{i}^{n}\right) \Delta_{n}$, which satisfies $\mathbb{E}\left(\left\|\zeta_{i, u}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}^{q}$ for $q \geq 2$.
Set

$$
\begin{aligned}
& \xi_{i}^{n}(1)=\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, u}^{n, l m} \zeta_{i, u}^{n, g h}, \quad \xi_{i}^{n}(2)=\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-2} \sum_{v=u+1}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m} \zeta_{i, v}^{n, g h} \\
& \xi_{i}^{n}(3)=\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-2} \sum_{v=u+1}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, g h} \text { and } \xi_{i}^{n}(4)=\frac{1}{k_{n}^{3} \Delta_{n}^{3}} \sum_{u=0}^{k_{n}-3} \sum_{v=u+1}^{k_{n}-2} \sum_{w=v+1}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m} \zeta_{i, w}^{n, g h} .
\end{aligned}
$$

The following bounds can be established,

$$
\begin{align*}
&\left|\mathbb{E}\left(\xi_{i}^{n}(1) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}  \tag{E.46}\\
&\left|\mathbb{E}\left(\xi_{i}^{n}(2) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}  \tag{E.47}\\
&\left|\mathbb{E}\left(\xi_{i}^{n}(3) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}  \tag{E.48}\\
&\left|\mathbb{E}\left(\xi_{i}^{n}(4) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}\right) . \tag{E.49}
\end{align*}
$$

## E.5.2 Proof of Equation (E.46)

The result readily follows from an application of the Cauchy Schwartz inequality coupled with the bound $\mathbb{E}\left(\left\|\zeta_{i+u}^{n}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{q}$ for $q \geq 2$.

## E.5.3 Proof of Equation (E.47)

Using the law of iterated expectation, we have, for $u<v$,

$$
\begin{equation*}
\mathbb{E}\left(\zeta_{i+u}^{n, j k} \zeta_{i+v}^{n, l m} \zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)=\mathbb{E}\left(\zeta_{i+u}^{n, j k} \mathbb{E}\left(\zeta_{i+v}^{n, l m} \zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i+u+1}^{n}\right) \mid \mathcal{F}_{i}^{n}\right) \tag{E.50}
\end{equation*}
$$

By successive conditioning, equation (E.43), and the Cauchy-Schwartz inequality, we also have

$$
\begin{aligned}
& \mid \mathbb{E}\left(\zeta_{i, v}^{n, l m} \zeta_{i v}^{n, g h} \mid \mathcal{F}_{i+u+1}^{n}\right)-\Delta_{n}^{2}\left(C_{i+u+1}^{n, l g} C_{i+u+1}^{n, m h}+C_{i+u+1}^{n, l h} C_{i+u+1}^{n, m g}\right) \\
& -\Delta_{n}^{2}\left(C_{i+u+1}^{n, g h}-C_{i}^{n, g h}\right)\left(C_{i+u+1}^{n, l m}-C_{i}^{n, l m}\right) \mid \leq K \Delta_{n}^{5 / 2}
\end{aligned}
$$

Given that $\mathbb{E}\left(\left|\zeta_{i+u}^{n, j k}\right|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq \Delta_{n}^{q}$, the approximation error involved in replacing $\mathbb{E}\left(\zeta_{i+v}^{n, l m} \zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i+u+1}^{n}\right)$ by $\Delta_{n}^{2}\left(C_{i+u+1}^{n, l g} C_{i+u+1}^{n, m h}+C_{i+u+1}^{n, l h} C_{i+u+1}^{n, m g}\right)+\Delta_{n}^{2}\left(C_{i+u+1}^{n, g h}-C_{i}^{n, g h}\right)\left(C_{i+u+1}^{n, l m}-C_{i}^{n, l m}\right)$ in equation (E.50) is smaller than $\Delta_{n}^{7 / 2}$.
We can also easily show that

$$
\begin{equation*}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l m}-C_{i+u}^{n, l m}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right) \tag{E.51}
\end{equation*}
$$

Since $\left(C_{i+u}^{n}-C_{i}^{n}\right)$ is $\mathcal{F}_{i+u}^{n}$-measurable, we use the successive conditioning, the Cauchy-Schwartz inequality, equation (E.42), equation (E.43), and the fifth statement in Lemma B2 applied to $Z=c$ to obtain

$$
\begin{align*}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, g h}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{5 / 2} \\
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k} \alpha_{i+u}^{n, l m}\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{5 / 2}  \tag{E.52}\\
\left.\left|\mathbb{E}\left(\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right)\right)\right| \mathcal{F}_{i}^{n}\right) \mid & \leq K \Delta_{n}
\end{align*}
$$

The following inequalities can be established using equation (E.42), the successive conditioning together with equation (B.10) for $Z=c$,

$$
\begin{aligned}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l g} C_{i+u+1}^{n, m h}+C_{i+u+1}^{n, l h} C_{i+u+1}^{n, m g}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{3 / 2} \\
\left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u+1}^{n, l g} C_{i+u+1}^{n, m h}+C_{i+u+1}^{n, l h} C_{i+u+1}^{n, m g}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{1 / 2} \\
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, g h}-C_{i}^{n, g h}\right)\left(C_{i+u+1}^{n, l m}-C_{i}^{n, l m}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right)
\end{aligned}
$$

The last three inequalities together yield $\left|\mathbb{E}\left(\xi_{i}^{n}(2) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}$.

## E.5.4 Proof of Equation (E.48)

First, note that, for $u<v$, we have

$$
\begin{equation*}
\mathbb{E}\left(\zeta_{i+u}^{n, j k} \zeta_{i+u}^{n, l m} \zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)=\mathbb{E}\left(\zeta_{i+u}^{n, j k} \zeta_{i+u}^{n, l m} \mathbb{E}\left(\zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i+u+1}^{n}\right) \mid \mathcal{F}_{i}^{n}\right) \tag{E.53}
\end{equation*}
$$

By successive conditioning and equation (E.42), we have

$$
\begin{equation*}
\left|\mathbb{E}\left(\alpha_{i+w}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i+v+1, w-v}\right) \tag{E.54}
\end{equation*}
$$

Using the first statement of Lemma applied to $Z=c$, it can be shown that

$$
\begin{aligned}
& \left.\left|\mathbb{E}\left(\left(C_{i+w}^{n, g h}-C_{i+v+1}^{n, g h}\right)\right)\right| \mathcal{F}_{i}^{n}\right)-\Delta_{n}(w-v-1) \widetilde{b}_{i+v+1}^{n, g h} \mid \\
\leq & K(w-v-1) \Delta_{n} \eta_{i+v+1, w-v} \leq K \Delta_{n}^{1 / 2} \eta_{i+v+1, w-v}
\end{aligned}
$$

The last two inequalities together imply

$$
\begin{equation*}
\left|\mathbb{E}\left(\zeta_{i+w}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)-\left(C_{i+v+1}^{n, g h}-C_{i}^{n, g h}\right) \Delta_{n}-\Delta_{n}^{2}(w-v-1) \widetilde{b}_{i+v+1}^{n, g h}\right| \leq K \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i+v+1, w-v}\right) \tag{E.55}
\end{equation*}
$$

Since $\underset{\sim}{\mathbb{E}}\left(\left|\zeta_{i, u}^{n, j k}\right| q \mid \mathcal{F}_{i}^{n}\right) \leq \Delta_{n}^{q}$, the error induced by replacing $\mathbb{E}\left(\zeta_{i+v}^{n, g h} \mid \mathcal{F}_{i+u+1}^{n}\right)$ by $\left(C_{i+v+1}^{n, g h}-C_{i}^{n, g h}\right) \Delta_{n}+\Delta_{n}^{2}(w-$ $v-1) \widetilde{b}_{i+v+1}^{n, g h}$ in equation (E.53) is smaller that $\Delta_{n}^{7 / 2}$.
Using Cauchy Schwartz inequality, successive conditioning, equation (E.52), equation (B.10) for $Z=c$ and the boundedness of $\widetilde{b}_{t}$ and $C_{t}$ we obtain

$$
\begin{aligned}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k} \alpha_{i+u}^{n, l m}\left(C_{i+u+1}^{n, j k}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i+u}^{n}\right)\right| & \leq K \Delta_{n}^{5 / 2} \\
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k} \alpha_{i+u}^{n, l m} \widetilde{b}_{i+u+1}^{n, g h} \mid \mathcal{F}_{i+u}^{n}\right)\right| & \leq K \Delta_{n}^{2}
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u+1}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{1 / 4} \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right) \\
\left|\mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \widetilde{b}_{i+u+1}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)\right| & \leq \Delta_{n}^{5 / 4} \\
\left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, g h}\right)\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \widetilde{b}_{i+u+1}^{n, g h} \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{1 / 2} \\
\left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u+1}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n} .
\end{aligned}
$$

The above inequalities together yield $\left|\mathbb{E}\left(\xi_{i}^{n}(3) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}$.

## E.5.5 Proof of Equation (E.49)

We first observe that $\xi_{i}^{n}(4)$ can be rewritten as

$$
\xi_{i}^{n}(4)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \zeta_{i+u}^{n, j k} \zeta_{i+v}^{n, l m} \zeta_{i+w}^{n, g h}
$$

where

$$
\begin{aligned}
& \zeta_{i+u}^{n, j k} \zeta_{i+v}^{n, l m} \zeta_{i+w}^{n, g h}=\left[\alpha_{i+u}^{n, j k} \alpha_{i+v}^{n, l m} \alpha_{i+w}^{n, g h}+\alpha_{i+u}^{n, j k} \Delta_{n} \alpha_{i+v}^{n, l m}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right)+\alpha_{i+u}^{n, j k} \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+w}^{n, g h}\right. \\
& \quad+\Delta_{n}^{2} \alpha_{i+u}^{n, j k}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right)+\Delta_{n}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right) \alpha_{i+v}^{n, l m} \alpha_{i+w}^{n, g h} \\
& \quad+\Delta_{n}^{2}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right) \alpha_{i+v}^{n, l m}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right)+\Delta_{n}^{2}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+w}^{n, g h} \\
& \left.\quad+\Delta_{n}^{3}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right)\right]
\end{aligned}
$$

Based on the above decomposition, we set

$$
\xi_{i}^{n}(4)=\sum_{j=1}^{8} \chi(j)
$$

with $\chi(j)$ defined below. We aim to show that $\left|\mathbb{E}\left(\chi(j) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}^{n}\right), j=1, \ldots, 8$. First, set

$$
\chi(1)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k} \alpha_{i+v}^{n, l m} \alpha_{i+w}^{n, g h} .
$$

Upon changing the order of the summation, we have

$$
\chi(1)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \alpha_{i+w}^{n, g h}
$$

Define also

$$
\chi^{\prime}(1)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \mathbb{E}\left(\alpha_{i+w}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)
$$

Note that $\mathbb{E}\left(\chi(1) \mid \mathcal{F}_{i}^{n}\right)=\mathbb{E}\left(\chi^{\prime}(1) \mid \mathcal{F}_{i}^{n}\right)$.
By Lemma B3, we have for $q \geq 2$,

$$
\mathbb{E}\left(\left\|\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{3 q / 4}
$$

The Cauchy-Schwartz inequality yields

$$
\begin{aligned}
& \mathbb{E}\left(\left.\left|\sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \mathbb{E}\left(\alpha_{i+w}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)\right|\right|_{i} ^{n}\right) \leq K k_{n}^{2}\left[\mathbb{E}\left(\left|\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right|^{4} \mid \mathcal{F}_{i}^{n}\right)\right]^{1 / 4} \\
& \times\left[\mathbb{E}\left(\left|\alpha_{i+v}^{n, l m}\right|^{4} \mid \mathcal{F}_{i}^{n}\right)\right]^{1 / 4} \times\left[\mathbb{E}\left(\left|\mathbb{E}\left(\alpha_{i+w}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)\right|^{2} \mid \mathcal{F}_{i}^{n}\right)\right]^{1 / 2} \leq K \Delta_{n} k_{n}^{2} \Delta_{n}^{3 / 4} \Delta_{n}^{3 / 2}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right),
\end{aligned}
$$

where the last iteration is obtained using equation (E.54) as well as the inequality $(a+b)^{1 / 2} \leq a^{1 / 2}+b^{1 / 2}$, which holds for positive real numbers $a$ and $b$, and the third statement in Lemma B1. It follows that

$$
\left|\mathbb{E}\left(\chi(1) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right)
$$

Next, we introduce

$$
\begin{aligned}
& \chi(2)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \Delta_{n}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\right) \alpha_{i+v}^{n, l m} \alpha_{i+w}^{n, g h} \\
& \chi(3)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+v}^{n, j k}\right) \Delta_{n}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+w}^{n, g h} \\
& \chi(4)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \Delta_{n}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\right) \Delta_{n}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+w}^{n, g h} .
\end{aligned}
$$

Given that for $q \geq 2$, we have

$$
\mathbb{E}\left(\left\|\sum_{u=0}^{v-1} \Delta_{n}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{3 q / 4} \text { and } \mathbb{E}\left(\left\|C_{i+u}^{n, j k}-C_{i}^{n, j k}\right\|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K_{q} \Delta_{n}^{q / 4}
$$

Similar steps to $\chi(1)$ lead to

$$
\left|\mathbb{E}\left(\chi(2) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right) \text { and }\left|\mathbb{E}\left(\chi(j) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right) \text { for } j=3,4
$$

Define

$$
\begin{aligned}
& \chi(5)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \Delta_{n}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right) \\
& \chi^{\prime}(5)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \Delta_{n} \mathbb{E}\left(\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i+v+1}^{n}\right) \\
& \chi(6)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \Delta_{n}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\right) \alpha_{i+v}^{n, l m} \Delta_{n}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right) \\
& \chi(7)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \Delta_{n}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right),
\end{aligned}
$$

where we have $\mathbb{E}\left(\chi(5) \mid \mathcal{F}_{i}^{n}\right)=\mathbb{E}\left(\chi^{\prime}(5) \mid \mathcal{F}_{i}^{n}\right)$. Recalling equation (E.55), we further decompose $\chi^{\prime}(5)$ as,

$$
\chi^{\prime}(5)=\sum_{j=1}^{5} \chi(5)[j]
$$

with

$$
\begin{aligned}
\chi^{\prime}(5)[1]= & \frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m}\left(\mathbb{E}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h} \mid \mathcal{F}_{i+v+1}^{n}\right)\right. \\
& \left.-\left(C_{i+v+1}^{n, g h}-C_{i}^{n, g h}\right) \Delta_{n}-\widetilde{b}_{i+v+1}^{n, g h} \Delta_{n}^{2}(w-v-1)\right) \\
\chi^{\prime}(5)[2]= & \frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \Delta_{n}\left(C_{i+v}^{n, g h}-C_{i}^{n, g h}\right)\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} \\
\chi^{\prime}(5)[3]= & \frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v+1}^{n, g h}-C_{i+v}^{n, g h}\right) \alpha_{i+v}^{n, l m} \\
\chi^{\prime}(5)[4]= & \frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}^{2}(w-v-1)\left(\widetilde{b}_{i+v+1}^{n, g h}-\widetilde{b}_{i+v}^{n, g h}\right) \alpha_{i+v}^{n, l m} \\
\chi^{\prime}(5)[5]= & \frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \Delta_{n}^{2}(w-v-1) \widetilde{b}_{i+v}^{n, g h}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \alpha_{i+v}^{n, l m} .
\end{aligned}
$$

Using equations (E.55), (E.54), and (E.51) and following the same strategy proof as for $\chi(1)$, it can be shown that

$$
\left|\mathbb{E}\left(\chi^{\prime}(5)[j] \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right), \text { for } j=1, \ldots, 5
$$

which in turn implies

$$
\left|\mathbb{E}\left(\chi(5) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right), \text { for } j=1, \ldots, 5
$$

The term $\chi(6)$ can be handled similarly to $\chi(5)$, hence we conclude that

$$
\left|\mathbb{E}\left(\chi(6) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\sqrt{\Delta_{n}}+\eta_{i, k_{n}}^{n}\right)
$$

Next, we set

$$
\chi(7)=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1}\left(\sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \Delta_{n}\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right)\right)
$$

Define

$$
\begin{aligned}
& \chi(7)[1]=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1}\left(\sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \Delta_{n}\left(C_{i+v+1}^{n, g h}-C_{i+v}^{n, g h}\right)\right) \\
& \chi(7)[2]=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1}\left(\sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \Delta_{n}\left(C_{i+v}^{n, g h}-C_{i}^{n, g h}\right)\right) \\
& \chi(7)[3]=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1}\left(\sum_{v=0}^{w-1}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right) \Delta_{n}^{2}(w-v-1)\left(\widetilde{b}_{i+v+1}^{n, g h}-\widetilde{b}_{i+v}^{n, g h}\right)\right)
\end{aligned}
$$

$$
\chi(7)[4]=\frac{1}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1}\left(\sum_{v=0}^{w-1} \Delta_{n}^{2}(w-v-1) \widetilde{b}_{i+v}^{n, g h}\left(\sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\right) \Delta_{n}\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right)\right)
$$

It is easy to see that

$$
\chi(7)=\sum_{j=1}^{4} \chi(7)[j]
$$

Similarly to calculations used for $\chi(1)$, it can be shown that

$$
\left|\mathbb{E}\left(\chi(7)[j] \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{1 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}\right), \text { for } j=1, \ldots, 3
$$

To handle the remaining term $\chi(7)[4]$, we decompose it $\chi(7)[4]=\sum_{j=1}^{9} \chi(7)[4][j]$, where

$$
\begin{aligned}
& \chi(7)[4][1]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l m}-C_{i+u}^{n, l m}\right)\left(C_{i+u+1}^{n, g h}-C_{i+u}^{n, g h}\right) \\
& \chi(7)[4][2]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l m}-C_{i+u}^{n, l m}\right) \\
& \chi^{\prime}(7)[4][2]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \mathbb{E}\left(\alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l m}-C_{i+u}^{n, l m}\right) \mid \mathcal{F}_{i+u}^{n}\right) \\
& \chi(7)[4][3]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, g h}-C_{i+u}^{n, g h}\right) \\
& \chi(7)[4][4]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \alpha_{i+u}^{n, j k} \\
& \chi(7)[4][5]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+u}^{n, j k}\left(C_{i+v}^{n, g h}-C_{i+u+1}^{n, g h}\right) \\
& \chi^{\prime}(7)[2][5]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right) \alpha_{i+u}^{n, j k} \mathbb{E}\left(\left(C_{i+v}^{n, g h}-C_{i+u+1}^{n, g h} \mid \mathcal{F}_{i+u}^{n}\right)\right. \\
& \chi(7)[4][6]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, l m}-C_{i+u}^{n, l m}\right)\left(C_{i+v}^{n, g h}-C_{i+u+1}^{n, g h}\right) \\
& \chi(7)[4][7]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \alpha_{i+u}^{n, j k}\left(C_{i+v}^{n, l m}-C_{i+u+1}^{n, l m}\right) \\
& \chi(7)[4][8]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\left(C_{i+u+1}^{n, g h}-C_{i+u}^{n, g h}\right)\left(C_{i+v}^{n, l m}-C_{i+u+1}^{n, l m}\right) \\
& \chi(7)[4][9]=\frac{\Delta_{n}^{2}}{\left(k_{n} \Delta_{n}\right)^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1} \alpha_{i+u}^{n, j k}\left(C_{i+v}^{n, l m}-C_{i+u+1}^{n, l m}\right)\left(C_{i+v}^{n, g h}-C_{i+u+1}^{n, g h}\right) .
\end{aligned}
$$

Using arguments similar to those involved for the treatment of $\chi(1)$, it can be shown that

$$
\left|\mathbb{E}\left(\chi(7)[4][j] \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{1 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}\right), \quad \text { for } j=1, \ldots, 8,
$$

which yields

$$
\left|\mathbb{E}\left(\chi(7) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{1 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}\right)
$$

Next, define

$$
\chi(8)=\frac{1}{k_{n}^{3}} \sum_{w=2}^{k_{n}-1} \sum_{v=0}^{w-1} \sum_{u=0}^{v-1}\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+w}^{n, g h}-C_{i}^{n, g h}\right) .
$$

This term can be further decomposed into six components. Successive conditioning and existing bounds give

$$
\begin{aligned}
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i+u}^{n, l m}\right)\left(C_{i+w}^{n, g h}-C_{i+v}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n} \\
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i+u}^{n, l m}\right)\left(C_{i+v}^{n, g h}-C_{i+u}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}\right) \\
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+v}^{n, l m}-C_{i+u}^{n, l m}\right)\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n} \\
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+w}^{n, g h}-C_{i+v}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n} \\
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+v}^{n, g h}-C_{i+u}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n} \\
& \left|\mathbb{E}\left(\left(C_{i+u}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+u}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+u}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}
\end{aligned}
$$

These bounds can be used to deduce

$$
\left|\mathbb{E}\left(\chi(8) \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}
$$

This completes the proof.

## E.5.6 Proof of Equations (B.13) and (B.14) in Lemma B5

Observe that

$$
\begin{aligned}
& \nu_{i}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)=\frac{1}{k_{n} \Delta_{n}} \sum_{u=0}^{k_{n}-1} \zeta_{i, u}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right), \\
& \nu_{i}^{n, j k} \nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)=\frac{1}{k_{n}^{2} \Delta_{n}^{2}} \sum_{u=0}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, u}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \\
& +\frac{1}{k_{n}^{2} \Delta_{n}^{2}} \sum_{u=0}^{k_{n}-2} \sum_{v=0}^{k_{n}-1} \zeta_{i, u}^{n, j k} \zeta_{i, v}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)+\frac{1}{k_{n}^{2} \Delta_{n}^{2}} \sum_{u=0}^{k_{n}-2} \sum_{v=0}^{k_{n}-1} \zeta_{i, u}^{n, l m} \zeta_{i, v}^{n, j k}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) .
\end{aligned}
$$

Hence, equations (B.13) and (B.14) can be proved using the same strategy as for (B.12).

## E.5.7 Proof of Equations (B.15) and (B.16) in Lemma B5

Note that we have

$$
\begin{aligned}
& \lambda_{i}^{n, j k} \lambda_{i}^{n, l m} \nu_{i}^{n, g h}=\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m}+\nu_{i}^{n, g h} \nu_{i}^{n, j k} \nu_{i}^{n, l m}-\nu_{i}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}-\nu_{i}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k} \\
& +\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)-\nu_{i}^{n, g h} \nu_{i}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)+\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right) \\
& -\nu_{i}^{n, g h} \nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)+\nu_{i}^{n, g h}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)
\end{aligned}
$$

and

$$
\lambda_{i}^{n, g h} \lambda_{i}^{n, j k} \lambda_{i}^{n, l m}=\nu_{i+k_{n}}^{n, g h} \nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m}+\nu_{i+k_{n}}^{n, g h} \nu_{i}^{n, j k} \nu_{i}^{n, l m}-\nu_{i+k_{n}}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}-\nu_{i+k_{n}}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}
$$

$$
\begin{aligned}
& +\nu_{i+k_{n}}^{n, g h} \nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)-\nu_{i+k_{n}}^{n, g h} \nu_{i}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)+\nu_{i+k_{n}}^{n, g h} \nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right) \\
& -\nu_{i+k_{n}}^{n, g h} \nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)+\nu_{i+k_{n}}^{n, g h}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right) \\
& -\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m}-\nu_{i}^{n, g h} \nu_{i}^{n, j k} \nu_{i}^{n, l m}+\nu_{i}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}+\nu_{i}^{n, g h} \nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k} \\
& -\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)+\nu_{i}^{n, g h} \nu_{i}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)-\nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right) \\
& +\nu_{i}^{n, g h} \nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)-\nu_{i}^{n, g h}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right) \\
& +\nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)+\nu_{i}^{n, j k} \nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)-\nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \\
& -\nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)+\nu_{i+k_{n}}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \\
& -\nu_{i}^{n, j k}\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)+\nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \\
& -\nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right)+\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) .
\end{aligned}
$$

From (A.4), notice that $\nu_{i}^{n}$ is $\mathcal{F}_{i+k_{n}}^{n}$-measurable and satisfies $\left\|\mathbb{E}\left(\nu_{i}^{n} \mid \mathcal{F}_{i}^{n}\right)\right\| \leq K \Delta_{n}^{1 / 2}$.
The law of iterated expectations and existing bounds imply

$$
\begin{align*}
\left|\mathbb{E}\left(\nu_{i}^{n, l m} \nu_{i+k_{n}}^{n, j k} \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{3 / 4}, \\
\left|\mathbb{E}\left(\nu_{i}^{n, l m} \nu_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k} \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}, \\
\left|\mathbb{E}\left(\nu_{i}^{n, l m}\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \nu_{i+k_{n}}^{n, j k} \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}, \\
\left|\mathbb{E}\left(\nu_{i+k_{n}}^{n, l m}\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n}^{3 / 4}, \\
\left|\mathbb{E}\left(\left(C_{i+k_{n}}^{n, j k}-C_{i}^{n, j k}\right)\left(C_{i+k_{n}}^{n, l m}-C_{i}^{n, l m}\right)\left(C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right) \mid \mathcal{F}_{i}^{n}\right)\right| & \leq K \Delta_{n} . \tag{E.56}
\end{align*}
$$

It can also be readily verified that

$$
\begin{aligned}
& \left|\mathbb{E}\left(\nu_{i+k_{n}}^{n, g h} \nu_{i+k_{n}}^{n, a b} \mid \mathcal{F}_{i+k_{n}}^{n}\right)-\frac{1}{k_{n}}\left(C_{i+k_{n}}^{n, g a} C_{i+k_{n}}^{n, h b}+C_{i+k_{n}}^{n, g b} C_{i+k_{n}}^{n, h a}\right)-\frac{k_{n} \Delta_{n}}{3} \bar{C}_{i+k_{n}}^{n, g h, a b}\right| \\
& \leq K \sqrt{\Delta_{n}}\left(\Delta_{n}^{1 / 8}+\eta_{i+k_{n}, k_{n}}^{n}\right)
\end{aligned}
$$

Hence, for $\varphi_{i}^{n, g h} \in\left\{\nu_{i}^{n, g h}, C_{i+k_{n}}^{n, g h}-C_{i}^{n, g h}\right\}$, which satisfies $\mathbb{E}\left(\left|\varphi_{i}^{n, g h}\right|^{q} \mid \mathcal{F}_{i}^{n}\right) \leq K \Delta_{n}^{q / 4}$ and $\mathbb{E}\left(\varphi_{i}^{n, g h} \mid \mathcal{F}_{i}^{n}\right) \leq$ $K \Delta_{n}^{1 / 2}$. One can show that

$$
\begin{aligned}
& \left|\mathbb{E}\left(\varphi_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)-\mathbb{E}\left(\left.\varphi_{i}^{n, g h}\left[\frac{1}{k_{n}}\left(C_{i+k_{n}}^{n, j l} C_{i+k_{n}}^{n, k m}+C_{i+k_{n}}^{n, j m} C_{i+k_{n}}^{n, k l}\right)-\frac{k_{n} \Delta_{n}}{3} \bar{C}_{i+k_{n}}^{n, j k, l m}\right] \right\rvert\, \mathcal{F}_{i}^{n}\right)\right| \\
& \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, 2 k_{n}}^{n}\right) .
\end{aligned}
$$

Next, by combining the successive conditioning together with existing bounds, we have

$$
\begin{aligned}
\left|\mathbb{E}\left(\varphi_{i}^{n, g h} \bar{C}_{i+k_{n}}^{n, j k, l m}\right)\right| & \leq K \Delta_{n}^{1 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, k_{n}}^{n}\right) \\
\left|\mathbb{E}\left(\varphi_{i}^{n, g h} C_{i+k_{n}}^{n, j l} C_{i+k_{n}}^{n, k m}\right)\right| & \leq K \Delta_{n}^{1 / 2},
\end{aligned}
$$

which together imply

$$
\begin{equation*}
\left|\mathbb{E}\left(\varphi_{i}^{n, g h} \nu_{i+k_{n}}^{n, j k} \nu_{i+k_{n}}^{n, l m} \mid \mathcal{F}_{i}^{n}\right)\right| \leq K \Delta_{n}^{3 / 4}\left(\Delta_{n}^{1 / 4}+\eta_{i, 2 k_{n}}^{n}\right) \tag{E.57}
\end{equation*}
$$

It is easy to see that equations (B.12), (E.56) and (E.57) and the inequality $\eta_{i, k_{n}}^{n} \leq \eta_{i, 2 k_{n}}^{n}$ together yield equations (B.15) and (B.16).

## E. 6 Proof of Lemma B6

Equation (B.17) can be proved easily using the bounds of $\rho(u, v)_{i}^{n, g h}$ in equation (E.41). To show equations (B.18), (B.19) and (B.20), we set

$$
\overline{\overline{A 11}}(H, g h, u ; G, a b, v)=\lambda(u, v)_{0}^{n} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right) \zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}
$$

Then,

$$
\frac{1}{\Delta_{n}^{1 / 4}}(\overline{\overline{A 11}}(H, g h, u ; G, a b, v)-\overline{A 11}(H, g h, u ; G, a b, v)) \stackrel{\mathbb{P}}{\Rightarrow} 0
$$

The above result is proved following similar steps as for equation (E.35) in case $w=1$ by replacing $\Theta(u, v)_{0}^{(C), i, n}$ by $\lambda(u, v)_{0}^{n}\left(\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right)-\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-2 k_{n}}\right)\right)$, which has the same bounds as the former. Next, decompose $\overline{\overline{A 11}}$ as follows,

$$
\begin{aligned}
\overline{\overline{A 11}}(H, g h, u ; G, a b, v) & =\lambda(u, v)_{0}^{n}\left[\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right) V_{i-1}^{n}\right. \\
& +\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right)\left(\mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)-V_{i-1}^{n}\right) \\
& \left.+\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right)\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b}-\mathbb{E}\left(\zeta(u)_{i}^{n, g h} \zeta(v)_{i}^{n, a b} \mid \mathcal{F}_{i-1}^{n}\right)\right)\right]
\end{aligned}
$$

We follow the proof of equation (E.36) for $w=1$, and we replace $\Theta(u, v)_{0}^{(C), i, n}$ by $\lambda(u, v)_{0}^{n}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right)$, which satisfies only the condition $\left|\lambda(u, v)_{0}^{n}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right)\right| \leq \widetilde{\lambda}_{u, v}^{n}$. This calculation shows that the last two terms in the above decomposition vanish at a rate faster than $\Delta_{n}^{1 / 4}$. Therefore,

$$
\frac{1}{\Delta_{n}^{1 / 4}}\left(\overline{\overline{A 11}}(H, g h, u ; G, a b, v)-\lambda(u, v)_{0}^{n}\left(\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right) V_{i-1}^{n}\right)\right) \Rightarrow 0
$$

As a consequence, for $(u, v)=(1,2)$ and $(2,1)$,

$$
\frac{1}{\Delta_{n}^{1 / 4}} \overline{\overline{A 11}}(H, g h, u ; G, a b, v) \Rightarrow 0
$$

The results follow from the following observation,

$$
\begin{aligned}
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\lambda(u, v)_{0}^{n}\left(\sum_{g, h, a, b=1}^{d} \sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right) V_{i-1}^{n}(u, v)\right)\right. \\
& \left.\quad-\frac{3}{\theta^{2}} \int_{0}^{T}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{t}\right)\left(C_{t}^{g a} C_{t}^{h b}+C_{t}^{g b} C_{t}^{h a}\right) d t\right) \Rightarrow 0, \quad \text { for }(u, v)=(2,2) \\
& \frac{1}{\Delta_{n}^{1 / 4}}\left(\sum_{g, h, a, b=1}^{d} \lambda(u, v)_{0}^{n}\left(\sum_{i=2 k_{n}}^{\left[T / \Delta_{n}\right]}\left(\partial_{g h} H \partial_{a b} G\right)\left(C_{i-1}\right) V_{i-1}^{n}(u, v)\right)-[H(C), G(C)]_{T}\right) \Rightarrow 0 \\
& \quad \text { for }(u, v)=(1,1)
\end{aligned}
$$

## F Additional Figures
















Figure F.1: Monthly $R^{2}$ of two Return Factor Models ( $\widehat{R}_{Y_{j}}^{2}$ ): the CAPM (the blue dotted line) and the Fama-French three factor model (the red solid line). Stocks are represented by tickers (see Table 1 for full stock names).


Figure F.2: Monthly $R^{2}$ of two Return Factor Models $\left(\widehat{R}_{Y j}^{2}\right)$ : the CAPM (the blue dotted line) and the Fama-French three factor model (the red solid line). Stocks are represented by tickers (see Table 1 for full stock names).


Figure F.3: Correlations between total and residual IdioVols: (a) $\operatorname{Corr}\left(C_{Z i}, C_{Z j}\right)$, (b) $\operatorname{Corr}\left(C_{Z i}^{r e s i d}, C_{Z j}^{r e s i d}\right)$ with one volatility factor, the market variance, (c) $\operatorname{Corr}\left(C_{Z i}^{r e s i d}, C_{Z j}^{r e s i d}\right)$ with ten volatility factors, the market variance and the variances of nine industry ETFs.


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[^1]:    ${ }^{1}$ A stock is said to be mispriced with respect to a given model if the expected value of the return on the stock is not consistent with the model.

[^2]:    ${ }^{2}$ The high frequency Fama-French factors are provided by Aït-Sahalia, Kalnina, and Xiu (2020).

[^3]:    ${ }^{3}$ In the Beta GARCH model, the IdioVol of a stock is a product of its own (total) volatility, and one minus the square of the correlation between the stock return and the market return.

[^4]:    ${ }^{4}$ Note that assuming that $Y$ and $C$ are driven by the same $d^{W}$-dimensional Brownian motion $W$ is without loss of generality provided that $d^{W}$ is large enough, see, e.g., equation (8.12) of Aït-Sahalia and Jacod (2014).
    ${ }^{5}$ The quadratic covariation of two vector-valued Itô semimartingales $X$ and $Y$, over the time span $[0, T]$, is defined as

    $$
    [X, Y]_{T}=p-\lim _{M \rightarrow \infty} \sum_{s=0}^{M-1}\left(X_{t_{s+1}}-X_{t_{s}}\right)\left(Y_{t_{s+1}}-Y_{t_{s}}\right)^{\top}
    $$

    for any sequence $t_{0}<t_{1}<\ldots<t_{M}=T$ with $\sup \left\{t_{s+1}-t_{s}\right\} \rightarrow 0$ as $M \rightarrow \infty$, where $p$-lim stands for the probability limit.

[^5]:    ${ }^{6}$ It is also possible to define more flexible kernel-based estimators as in Kristensen (2010).

[^6]:    ${ }^{7}$ Note that $\widetilde{\sigma}_{s}=\left(\widetilde{\sigma}_{s}^{g h, m}\right)$ is $\left(d \times d \times d^{W}\right)$-dimensional and $\widetilde{\sigma}_{s} d W_{s}$ is $(d \times d)$-dimensional with $\left(\widetilde{\sigma}_{s} d W_{s}\right)^{g h}=$ $\sum_{m=1}^{d^{W}} \widetilde{\sigma}_{s}^{g h, m} d W_{s}^{m}$.

[^7]:    ${ }^{8}$ For the $j^{\text {th }}$ stock, our analog of the coefficient of determination in the R-FM is $R_{Y j}^{2}=1-\frac{\int_{0}^{T} C_{Z j, t} d t}{\int_{0}^{T} C_{Y j, t} d t}$. We estimate $R_{Y j}^{2}$ using the general method of Jacod and Rosenbaum (2013). The resulting estimator of $R_{Y j}^{2}$ requires a choice of a block size for the spot volatility estimation; we choose two hours in practice (the number of observations in a block, say $l_{n}$, has to satisfy $l_{n}^{2} \Delta_{n} \rightarrow 0$ and $l_{n}^{3} \Delta_{n} \rightarrow \infty$, so it is of smaller order than the number of observations $k_{n}$ in our estimators of Section 3).
    ${ }^{9}$ Our measure of correlation between the idiosyncratic returns $d Z_{i}$ and $d Z_{j}$ is

    $$
    \begin{equation*}
    \operatorname{Corr}\left(Z_{i}, Z_{j}\right)=\frac{\int_{0}^{T} C_{Z i Z j, t} d t}{\sqrt{\int_{0}^{T} C_{Z i, t} d t} \sqrt{\int_{0}^{T} C_{z j, t} d t}}, \quad i, j=1, \ldots, d_{S}, \tag{40}
    \end{equation*}
    $$

[^8]:    where $C_{Z i Z j, t}$ is the spot covariation between $Z_{i}$ and $Z_{j}$. Similarly to $R_{Y j}^{2}$, we estimate $\operatorname{Corr}\left(Z_{i}, Z_{j}\right)$ using the method of Jacod and Rosenbaum (2013).
    ${ }^{10}$ We also considered the volatility of size and value Fama-French factors. However, both these factors turned out to have very low volatility of volatility and therefore did not significantly change the results.

[^9]:    ${ }^{11}$ Recall that we are using false discovery rate to control for multiple testing, and notice from Figure 2 that the number of individually rejected null hypotheses is 16 , less than $0.05 \times 435$.

[^10]:    ${ }^{12}$ The Feller property is satisfied implying the positiveness of the processes $\left(f_{j, t}\right)_{1 \leq j \leq 4}$.
    ${ }^{13}$ Notice that by Itô Lemma, each of these three models can be expressed at the level of equation (1) for the vector

[^11]:    Table 5: Finite sample properties of our estimators using 10 years of data sampled at 5 minutes. The true values are $\gamma_{Z 1}=0.450, R_{Z 1}^{2, \text { IdioVol-FM }}=$

[^12]:    Table 6: Finite sample properties of our estimators using 10 years of data sampled at 1 minute. The true values are $\gamma_{Z 1}=0.450, R_{Z 1}^{2, \text { IdioVol-FM }}=0.336$, $\operatorname{Corr}\left(C_{Z 1}, C_{Z 2}\right)=0.514, \operatorname{Corr}\left(C_{Z 1}^{\text {resid }}, C_{Z 2}^{\text {resid }}\right)=0.408$. Model 3.

